

XXX. *M. DE LUC's Rules, for the measurement of Heights by the Barometer, compared with Theory, and reduced to English Measures of Length, and adapted to Fahrenheit's Scale of the Thermometer : with Tables and Precepts, for expediting the practical Application of them. By Samuel Horsley, LL.D. addressed to Sir John Pringle, Bart. P. R. S.*

TO SIR JOHN PRINGLE, BART. P. R. S.

SIR,

Redde, Feb. 17, 1774. **I**N the papers which I have the honor to present to you, nothing more was at first intended, than to reduce the *formulae* given by Mr. DE LUC, in his elaborate work upon *the modifications of the atmosphere*, for expressing differences of elevation, as indicated by the barometer and thermometer, in Paris toises, to others, which, from the like *data*, should express such differences in English fathom ; and at the same time be adapted to that scale of the thermometer, which is in general use in this country. Had I confined myself to this, which was my original

ginal design, I should certainly have suppressed whatever I might have executed, as I find my learned friend the ASTRONOMER ROYAL hath devoted some of his leisure hours to these calculations; and the reductions, I proposed, are actually performed, in a short and elegant paper of his upon the subject, which is already, I believe, in Dr. MATY's hands. But I had made but a small progress in my intended work, when it occurred to me, that *tables of the equations* for the effects of heat and cold, would be very useful. Such I have taken the pains to construct, and have carried them to as great a length, as can be ever wanted. They are annexed to the ensuing piece, and will render the application of Mr. DE LUC's rules more easy and expeditious, than any peculiar divisions of the thermometrical scale. When my tables were finished, I thought it might be still further useful, to give a succinct explanation of Mr. DE LUC's original *formulae*; that such as have not leisure to peruse his excellent work, might be furnished with a competent idea of the result of his researches. This, I found, I could not do, in any way so satisfactory to myself, as by opening, as I went along, the principles of theory, in which the conclusions, he hath arrived at, appear to originate. And thus I was insensibly led into minute disquisitions, concerning the agreement of Mr. DE LUC's conclusions, from a long train of accurate experiments, with the geometrical theory of the atmosphere, founded on the general laws of gravitation. In this manner, SIR, the papers before you have taken, as it were spontaneously, the form, in which they now appear; and a subject, remotely connected
with

with my first design, takes up the far greater part of them. They can hardly be free from the imperfections naturally incident to productions, in which the plan hath been gradually changed, during the actual progress of the work. I flatter myself, however, that they are not deficient in two essential points, the precision and the perspicuity of the mathematical reasoning; and that, however unfinished in some particulars, they are such upon the whole, as the dignity of the subject may, in some degree, commend to your protection. There is perhaps no branch of physical enquiry, intrinsically more sublime and interesting, nor likely to be more important in its uses, than that which immediately regards the constitution of that elastic fluid, which surrounds our globe, and appears to be a principal agent, in many of the most striking operations of nature, and a necessary instrument, at least, in carrying on the wonderful business of vegetation and of animal life. Upon a subject of so much importance, it must afford satisfaction, to find so exact an agreement, as is evinced, if I am not much deceived, in the ensuing pages, between a multiplicity of experiments, not suggested by any previous speculations of theory, with a theory, whose conclusions in this branch had never before been duly submitted to the test of experiment.

The whole of the following paper is divided into six sections. The business of the first, is merely preparation for the principal work. It contains a brief account of the sum of Mr. DE LUC's researches concerning the *variation of the heat of boiling*

boiling water, so far only as they respect the construction and comparison of mercurial thermometers; and it exhibits an actual comparison of the scale of M. DE LUC's thermometer, with that which is likely to continue in general use here. In the second, I state the general principles of measuring heights by the barometer; or in other words, of the modification of the air, with respect to its condensations, at different heights, by the pressure of the superior parts, exclusive of every other cause. These principles are recapitulated for the sake of perspicuity and order; but, as they are generally well known, I refer to former writers for the demonstration of them. This section concludes with the exposition of M. DE LUC's fundamental rule. The third section treats of the equation depending on the difference of temperature of the quicksilver in the barometers, at the different places of synchronous observation. It explains, whence the necessity of this correction arises, and considers a case, in which the application of it requires some particular attentions; namely, the levelling a tract of flat country by the barometer. The fourth section explains the equation depending on the temperature of the air. This is likewise traced to its origin in theory; and upon this occasion, I enquire into the condensations of a fluid, unequally elastic in its different parts, under different degrees of compressive force: The fifth section contains the reduction of M. DE LUC's *formulæ*. I have found, with pleasure, by the perusal of Mr. MASKELYNE's paper, that we agree in our final conclusions. To this section, I have subjoined two problems; the one for esti-

mating the variations which the density of the air, at any place, undergoes ; and the other for determining the specific gravity of the air of any temperature, at any elevation. The sixth and last section points out certain consequences, which seem to follow from the discoveries already made in conjunction with the theory established. These are only propounded to awaken curiosity, and promote enquiry. I am well aware, how little theory is to be trusted, in its *remote* conclusions, on account of the necessary deficiencies of the physical *data*, upon which its reasonings are founded. The true uses of it are, either to explain the mutual connexions and dependencies of things already known, or to suggest *conjectures* concerning what is unknown, to be tried by future experiment. And he who applies it, with due circumspection, to these purposes, will always find it an useful engine. I flatter myself, that I have assigned the true cause of some very singular phænomena, remarked by M. DE LUC. I have endeavoured, to treat every part of my subject, in the plainest manner, that the nature of it would admit ; and so diffusely, as to be, I hope, intelligible to all, who are moderately well founded in the mathematical sciences ; for I have observed with pleasure, that M. DE LUC's book hath raised a general curiosity upon the points it treats of ; and for that reason, it seemed the more necessary, to depart in this instance, from a practice of late become too general, with mathematicians, *to write only for one another*. To which, if I am not mistaken, it is in great measure owing, that these noble and useful studies are less generally attended to, than they were in former times ; when men of eminence,

eminence, in this, as well as in all other branches, placed their principal glory, no less in communicating knowledge, than in acquiring, or seeming to possess it; and were content to dedicate large portions of their time, to the removing of those difficulties for others, which they had once surmounted for themselves. It is true, that in the variety of mathematical discussions, there are some of such a nature, that it would be a desperate attempt to render them intelligible to any but mathematicians; to those, who to the natural faculty of combining, have added the mechanical and acquired habits of analytic calculation. Such subjects are best treated with scientific brevity. But this is no reason for wrapping up others in a similar stile; and I must observe, that *conclusions*, at least, are always susceptible of simple perspicuous exposition, however abstruse the mode of investigation, in some cases, may necessarily be, by which they are brought out.

Such, SIR, is the general scope and plan of the following piece, which is presented to you, in testimony of the author's respect and gratitude.

I have the honour to be, SIR,

your much obliged,

and most obedient servant,

Temple,
Nov. 18, 1773.

SAMUEL HORSLEY.

SECTION FIRST.

Of the VARIATION of the point of BOILING WATER, and
the COMPARISON of THERMOMETERS.

THE degree of heat wherewith any fluid boils, is invariably the same, under a given pressure; but if the pressure be diminished or increased, the boiling heat is diminished or increased.

Water, placed under the exhausted receiver, would be converted into steam, with a degree of heat, far inferior to that, which is necessary to its boiling in the open air; and under the pressure of its own vapour, confined in PAPIN's digester, it is said to sustain a degree of heat, without boiling, far exceeding that, which, in the open air, would convert it into steam.

Hence it follows, that in climates, where the pressure of the atmosphere is liable to considerable change, the heat of boiling water, in the open air, will be different, at different times. Consequently, thermometers, made in different states of the barometer, will disagree; unless allowance hath been made, for the effect of the variation of the barometer, upon accurate principles.

If care were taken to adjust the boiling point, to the mean height of the barometer, in every country, the instruments of the same country would always be consistent; but those of different countries would still disagree; that is, they would express the same temperature differently, though their fundamental intervals should be similarly divided; for, in every scale, the number of degrees above

or below melting ice, by which any given temperature is expressed, will be as the value of each degree inversely; that is, if each be a given part of the fundamental interval, as the value of the fundamental interval inversely; but, if the degrees of different scales be different parts of the fundamental intervals, as the value of the fundamental interval inversely, and the number of degrees contained in it directly.

It is necessary here to explain some of my expressions. By the *fundamental interval* is to be understood the whole extent of the scale between melting ice and boiling water. This is not a particular *length*; for its length is not the same, even in thermometers made in the same state of the barometer, unless their figures be equal and similar; but it is a portion of the whole *solid content* of the thermometer, including so much of the tube, as reaches up to the point of melting ice; and it is the same portion of the whole solid content in all made in the same state of the barometer; but in such as have been made in different states, a different portion in each. The degrees of the scale are subdivisions of this portion of solid content, into lesser aliquot parts. With respect to the number of these subdivisions, the practice of different countries is different. By the *value* of the fundamental interval, or of the degrees of a thermometer, I mean the fractions, which express what parts they are respectively of the whole solid content, terminated as above; that is, by the point of melting ice: and by the proportion of the fundamental intervals, or of the degrees, of different instruments, compared together, I understand the proportion of these fractions.

tions. Or every thing may be reduced to a comparison of lengths, if in comparing the instruments of different countries, the comparison be imagined to be made between two of equal dimensions, and similar figure, setting aside that part of each which is above the point of melting ice.

To compare the thermometers, therefore, of different countries, the proportions of their fundamental intervals to each other must be ascertained; or, we must have some means of finding, upon one scale, the place of the boiling point of another. For this purpose, a general solution is requisite of the following problem: "*The fundamental interval being given for a given height of the barometer, to find the fundamental interval, for any other given height of the barometer.*" The solution is furnished by M. DE LUC's laborious researches.

M. DE LUC fixes the boiling point of his thermometer, when the barometer is at 27 inches French ^(a), that being its mean height at GENEVA. He divides the fundamental interval, after the French manner, into 80 equal parts; and, by a great number of experiments, on the heat of boiling water, at different heights above the level of the sea, the detail of which is to be found in his *Essai sur la Variation*, &c: he hath found, that the height of his thermometer, plunged in boiling water, may be expressed, in all states of the barometer, by the following formula, viz. $\frac{2}{3} \frac{2}{3} \frac{2}{3} \log. y - a = T$. In which, y denotes the height of the barometer, in sixteenths of a Paris line: T the height of a thermometer, plunged in boiling water, above melt-

(a) Recherches, sur les Modificat. de l'Atmosphere, §. 451. a.
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ing ice, in hundredths of a degree of M. DE LUC scale; and a the constant number 10387 ^(b).

It is proper in this place to inform the reader, that M. DE LUC, by logarithms, always means the tabular or Briggian logarithms, and considers the 7 figures given by the tables, besides the index, as integral figures; that is, he considers the eighth figure of the logarithm as standing in the place of units. Throughout this paper, I have conformed myself to this manner of conceiving the tabular logarithms, that, upon the same subject, we may speak the same language. But it is more usual, with mathematicians, and, in general, it is more convenient, to consider all the figures, after the index, as decimals. Thus the number, which M. DE LUC expresses by $\frac{99}{200000} \log y$, would, in the common mathematical stile, be $\frac{99 \times 100}{2} \log y$; or, $99 \times 50 \log y$.

It is but seldom that the barometer in this country stands so low as 27 French inches. 30 inches English are little more than its mean height upon the

(b) Ibid. §. 961. and §. 1143. note a . It may seem doubtful whether there is not a small error in the constant number 10387. The experiments for ascertaining the variation of the boiling point were made with a thermometer, of a peculiar scale; and the formula deduced from them was this, $\frac{1133}{200000} \log y - 5015$ = the height of the thermometer, plunged in boiling water, above melting ice, in parts of that peculiar scale. Recherch. sur les Modificat. de l'Atmosphere, §. 960. Now a degree of this scale was afterwards found to be to a degree of M. DE LUC's common scale as 80 to 3869 $\frac{1}{4}$. And this proportion between the degrees of the two scales, gives 10368,9 or 10369 very nearly, instead of 10387, for the value of the constant number a , in the formula for hundredths of a degree of the common scale.

plane country in the neighbourhood of LONDON. It may, therefore, be proper for the London workmen to fix their boiling point, when the barometer is at 30 inches. FAHRENHEIT's division of the scale, which makes 180 degrees between melting ice and boiling water, and places the point of 0 at the 32d degree below melting ice, may be retained; because long use hath rendered it familiar to us. A thermometer thus constructed, I shall call BIRD's FAHRENHEIT: that eminent mechanic, our countryman, Mr. JOHN BIRD, having been, as I believe, the first workman, who took the pains to attend to the state of the barometer, in making thermometers, and having always made it his practice, to fix the boiling point, when his barometer hath stood at 30 inches. Taking it for granted, that this scale will continue in general use here, I shall give an accurate comparison of it with the scale of M. DE LUC.

T being put for the height of a thermometer plunged in boiling water, above melting ice, in hundredths of a degree of M. DE LUC's scale, in any given state of the barometer; let Θ denote the same height, in hundredths of a degree of BIRD's Fahrenheit.

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|---|------------------------------|
| Put y for the height of the barometer | in 16ths of a Paris line. |
| v for its height | in Paris lines. |
| x | in 10ths of a Paris inch. |
| z | in 10ths of an English inch. |

And for 10387 put a .
 for 16 put b .
 for 10 put c .
 for 12 put d .

Also put E and F for numbers expressing the proportion of the English foot to the French foot.

Now

Now M. DE LUC hath found, that whatever be the value of y ;

$$\frac{9999}{10000} \log. y - a = T.$$

$$\text{But } \log. y = \log. v + \log. b.$$

$$\text{and } \log. v = \log. x + \log. d - \log. c.$$

$$\text{and } \log. x = \log. z + \log. E - \log. F.$$

$$\text{Therefore } \log. y = \log. z + \log. E + \log. d + \log. b. - \log. F - \log. c.$$

$$\text{and } \frac{9999}{10000} \log. z + \frac{9999}{10000} \log. E. + \log. d + \log. b. - \log. F - \log. c - a = T.$$

$$\text{But } \frac{9999}{10000} \log. E. + \log. d + \log. b. - \log. F - \log. c - a = -4171.55.$$

the French foot being to the English as 2,1315 to 2; *vide* Phil. Transf. vol. LVIII.

$$\text{Therefore } \frac{9999}{10000} \log. z - 4171.55 = T.$$

And $\frac{9999}{10000} \log. z - 41,7155 = \frac{T}{100} =$ the height of the thermometer, plunged in boiling water, above melting ice, in degrees of M. DE LUC's scale, when the height of the barometer, in tenths of an English inch, is z .

Now for z write 300. Then $\frac{T}{100} = 80,902$. which therefore is the height of the thermometer, in boiling water, above melting ice, in degrees of DE LUC's scale, when the barometer is at 30 inches English. And in the same state of the barometer, the height of the thermometer plunged in boiling water, above melting ice, in degrees of BIRD'S Fahrenheit, or, $\frac{\Theta}{100}$, is 180. Hence the numbers T and Θ are in the constant proportion of 809 and 1800, whatever be the value of z . For the change produced in the heat of boiling water, by any change of z , being always the same for both thermometers, the temperature expressed by T in parts of one scale, is always the same,

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same, as Θ expresses in parts of the other; and, therefore, putting $\frac{1}{L}$, and $\frac{1}{B}$, for the values of the hundredth part of a degree, of the scales of M. DE LUC and BIRD respectively, the fractions $\frac{T}{L}$, $\frac{\Theta}{B}$ are always equal, and T , Θ , are in the constant proportion of the invariable numbers L , B : consequently, when the proportion of T and Θ is determined for any particular value of z , it is found generally for all. Therefore, as was affirmed,

$$T : \Theta = 809 : 1800.$$

And $T = \frac{809}{1800} \Theta = \frac{809}{20000} \Theta$ very nearly (c) in all values of z , and substituting this value, for T , in the equation exhibiting the relation between z and T , we shall have, for the relation between z and Θ ,

$$\frac{99}{20000000} \log. z - 41,7155 = \frac{899}{2000 \times 100} \Theta.$$

Or, $\frac{99}{10000 \times 899} \log. z - 92,804 = \frac{\Theta}{100}$ = the height of the thermometer in boiling water, above melting ice, in degrees of BIRD's Fahrenheit, when the height of the barometer, in 10ths of an English inch, is z . And thus M. DE LUC's *formula*, for the variation of the boiling point, is adapted to English instruments, and reduced to English measures of length.

(c) It might be sufficiently accurate for most purposes, to substitute $\frac{2}{3} \Theta$ ($= \frac{811}{1800} \Theta$) for $\frac{809}{1800} \Theta$. The error of this substitution would be about $\frac{1}{1800} \Theta$; and consequently would amount to about $\frac{2}{3}$ of 1° of BIRD's Fahrenheit, when z is 300. But the error in the substitution I have used is much less, not amounting to $\frac{1}{18000} \Theta$, which makes less than $\frac{1}{44}$ of a degree of BIRD's scale, in the same case.

For z write 287,7525 (the length of 27 French inches in tenths of an English inch) and $\frac{\ominus}{100}$, the height of DE LUC's boiling point, above melting ice, in degrees of BIRD's Fahrenheit, comes out 177,989. Hence M. DE LUC's boiling point falls upon 209,989 of BIRD's scale; that is, upon 210 very nearly, or insensibly more than two degrees below BIRD's point of boiling; and the reduction of either scale to the other, in all inferior temperatures, will be as the table of comparison shews.

By M. DE LUC's *formula*, thus reduced, the height of the thermometer, plunged in boiling water, above melting ice, in degrees of BIRD's Fahrenheit, in any given state of the barometer, may be computed. But 899 being a troublesome divisor, to render the computation more easy and expeditious, take the following method. For $\frac{1}{899} \log. z$. write s .

Then $s + \frac{1}{899} s - 92,804 = \frac{\ominus}{100}$ very nearly. ^(d)

Upon these principles I have computed a little table, for finding the heights, to which a good BIRD's

(d) If, according to note (b), we take 10369 instead of 10387, for the value of the constant number a in M. DE LUC's *formula*, we shall find, $\frac{1}{899} \log. z - 41,5355 = \frac{T}{100}$.

Whence we should obtain $s - \frac{1}{899} s - 92,198 = \frac{\ominus}{100}$ very nearly. But I abide by the *formula* given in the text: being persuaded, that M. DE LUC hath purposely adopted the number 10387, as agreeing better, upon the whole, with his experiments than the other; though I do not recollect that he hath, in any part of his work, expressly said so.

Fahrenheit will rise, when plunged in boiling water, in all states of the barometer, from 27 to 31 inches English. Among other uses of this table, it will serve for a direction to instrument-makers, to make a true allowance, for the effect of the variation of the barometer, if they are at any time obliged, to finish a thermometer, when the barometer is above or below 30 inches ; but in general it should be their rule, to watch an opportunity of fixing the boiling point, when the barometer is actually at the height prescribed.

I must, upon this occasion, declare, how heartily I concur with M. DE LUC, in wishing, that some *common* scale, the same in the number of its divisions, its point of 0, and its boiling point, might be received, with unanimous consent, by philosophers of all parts of the world ; that, for the future, we might have one general language, for so very general an object of enquiry and discourse, as the different degrees of heat and cold. To mathematicians the comparison of different scales, is a task of little labour. But among those who have a taste for physical researches, and are capable of pursuing them, in certain branches at least, with some degree of advantage to science, as well as to themselves, there are many, to whom every little calculation is a toil. Not having acquired the habits of it, in the early part of their lives, they never can acquire them, in such a degree, as to be able to put a confidence in themselves, and rely upon the accuracy of their own conclusions. The ease of such persons should, in my opinion, be consulted. But these are not the only persons interested. The inconveniencies to be apprehended from a diversity of
scales

scales are general. It is not one of the least, that instruments essentially different, as made at different elevations in the atmosphere, will continue to pass under the same name. The error and confusion, which this may create, is remarkably instanced, in what hath actually happened to the thermometer of the celebrated REAUMUR. The instrument, which at this day passes, all over Europe, under the name of REAUMUR's thermometer, is essentially different from his; yet it was always supposed to be the same, even among his own countrymen, till M. DE LUC detected the mistake. At what time, upon what occasion, or by whom, the change was introduced, is still unknown.

SECTION SECOND.

Of the GENERAL PRINCIPLES of measuring HEIGHTS by the
BAROMETER.

AS it is my design to compare the practical rules, which M. DE LUC hath deduced from experiment, with theory, it is necessary, for the clearer arrangement of the argument, previously to state the general principles, upon which the measurement of heights by the barometer depends. In doing this, I shall rather aim at perspicuity than at brevity; referring, however, for the demonstration of whatever hath been advanced before, to writers of approved authority.

In any column of the atmosphere, resting perpendicularly upon any small portion of the earth's surface, the densities of the air diminish, as we ascend to greater heights; and if the accelerative force of gravity were the same

at.

at all heights, the densities would decrease geometrically as the height increased arithmetically. This is an obvious consequence from a known property of air, that it expands itself through a greater or a less space, in proportion as the force, by which it is compressed, is less or greater; that is, that the density of air, is always as the compressing force (*e*). And from hence it would follow, that the difference of the elevation of any two places, would be as the logarithm of the *ratio* of the densities of the air, at each: and the density being every where as the compressing force, and the compressing force as the length of the column of quicksilver sustained by it in the barometer, the difference of elevation would be, as the logarithm of the *ratio* of the altitudes of the quicksilver in the barometer, at the same time, at the different stations; that is, as the difference of the tabular logarithms of the numbers, by which those altitudes would be expressed in any given measure (*f*). But the accelerative force of gravity diminishes, in the same proportion, as the square of the distance, from the earth's center, is increased. It is not the same

(*e*) Cotes's Hydrostat. Lectures, lect. ix.

(*f*) See Philosoph. Transact. n. 181. Phil. Nat. Princip. Math. lib. ii. prop. 22. Scholium. Cotes's Hydrostat. Lectures, lect. ix. Harmon. Mens. p. 17.

Of all these demonstrations, that given by Mr. COTES, in the hydrostatical lectures, will be the most perspicuous to the generality of readers. It is very diffuse, and he hath been at great pains to reduce it to the most simple principles. Mathematicians will find the substance of his argument well summed up, by M. DE LA LANDE, in the Connoissance for the year 1765, p. 211, 212. who, I am persuaded, would not have produced it as a new demonstration, had he known it had been given before.

therefore

therefore at all heights above the surface; and the densities of the air must decrease, by a different law, from that which would obtain, if the force of gravity were uniform. This other law, however, is such, that, to a much greater height than is accessible to man, the gradual variation of the compressing force and densities of the air, will be so little different, from what it would be upon the former hypothesis, that the error of that hypothesis, in the measurement of heights, will be absolutely insensible.

Let the point C [tab. IX. fig. 1.] represent the center of the earth. CA the earth's semi-diameter. AB any height above the surface. At A, place a right line, AD, of any finite length, at right angles with AC. In the right line AC, towards C, take A β , such that CA may bear to A β the proportion of CB to BA. In a right line drawn through β , at right angles with AC, take β E, of such length, as to bear to AD the proportion of the density of the air at B to the density at A, or at the earth's surface. The curve, which the point E always touches, is a logarithmic, of which AC is the asymptote (^g).

As I shall have frequent occasion to consider the curve, which thus exhibits the relation between density and elevation, I shall call it the ATMOSPHERICAL LOGARITHMIC.

Imagine this curve described, and take another height Ab, and take $A\epsilon = \frac{CA \times Ab}{Cb}$, and draw C ϵ

(^g) Cotes's Hydrostat. Lectures, p. 161—167. Harmon. Mens. p. 18. Phil. Nat. Princip. Math. lib. ii. prop. 22. Brooke Taylor. Method. Increment, Prop. 26.

parallel to βE , meeting the curve in e . Then βe is the logarithm of the *ratio* of βE to $e e$, or of the density at B to the density at b . But if the greater of the two heights, AB and Ab, bear but a very small proportion to the semi-diameter of the earth, their difference Bb will be very nearly equal to βe .

For, because $CB : BA = CA : A\beta$ (by construction.)

Therefore, by conversion, $CB : CA = CA : C\beta$.

In like manner, and by inversion, $CA : Cb = C e : CA$.

by equi-distance perturbate, $CB : Cb = C e : C\beta$.

and converting, $CB : Bb = C e : \beta e$.

by permutation, $Bb : \beta e = CB : C e$.

But when AB is infinitely diminished, $CB = CA$ ultimately. Also Ab being infinitely diminished, $C e = CA$ ultimately. Therefore $CB = C e$ ultimately, and $Bb = \beta e$ ultimately. Q. E. D.

Now AB and Ab will always be so small, with respect to CA, if B and b be supposed to represent any accessible places, that CB, C e, and Bb, βe , may always, in this case, be considered, as in their ultimate proportion of equality.

It is still therefore to be admitted as a principle, in practice, that the difference of elevation of any two places, is as the difference of the tabular logarithms of the heights of the quicksilver in the barometer at the same time, at both places; that is, it is the logarithm of the *ratio* of those heights in *some* system of logarithms. And the heights of the quicksilver being given, by observation, the difference of elevation will be known, if that particular system can be determined; that is, if the *modulus* of the system, or the length of the subtangent of the curve

curve DEe of that system, can be ascertained, in some known measure, as English fathoms, or Paris toises.

The easiest method of doing this, that theory suggests, is to compare barometers at two stations, suppose B and b , each of a known elevation AB and Ab , above the level of the sea. For the logarithms of any given *ratio*, in different systems, are proportional to the subtangents; and the difference of elevation, Bb , diminished in the proportion of CB , (the distance of the higher station from the earth's center) to Cc , (a third proportional to Cb , the distance of the lower station from the earth's center, and CA , the earth's semi-diameter) is the logarithm of the *ratio* of the density at B , to the density at b (that is, of the columns of quicksilver sustained in the barometer at B and b) in the atmospherical system. Therefore, as the difference of the tabular logarithms, of these columns, to the subtangent of the tabular system, so should Bb , diminished as hath been said, (that is, so should βc) be to the subtangent of the atmospherical logarithmic. The utmost height, to which we can ascend, above the level of the sea, is so small, that the reduction of Bb may, even in this investigation, always be neglected. For, if AB were four English miles, which exceeds the greatest accessible heights, even of the Peruvian mountains, and Ac three, βc would be scarce one part in 500 less than Bb . So that, by comparing barometers at different elevations, within a mile above the level of the sea, the subtangent of the atmospherical curve might be determined, as it should seem, without sensible error, by taking simply the difference of elevation, without reduc-

tion, for the logarithm of the *ratio* of the observed heights of the quicksilver in the atmospherical system. Those, however, who have attempted to determine the system, by this method, have hitherto agreed but ill in their conclusions. The fact is, that the length of this subtangent is very different at different times. The causes and quantity of its variation will be considered in another place. It appears, from M. DE LUC's experiments, that, though otherwise subject to change, it is constant in a given temperature. And that when the temperature of the air is $+16\frac{3}{4}$ of his scale, the difference of the tabular logarithms of the heights of the quicksilver in the barometer, gives the difference of elevation in 1000ths of a Paris toise ^(b). This is the rule, which he hath derived from a great number of experiments made at very different elevations: and the truth of it being admitted, it is a necessary consequence, that the number, which is the *modulus* of Briggs's system, expresses the length of the subtangent of the atmospherical curve, such as it is in that temperature, in 1000ths of a Paris toise.

(b) *Recherch. sur les Modif. de l'Atmosph.* §. 588.

SECTION THIRD.

Of the EQUATION for the DIFFERENCE of TEMPERATURE of the QUICKSILVER in the BAROMETERS, at the DIFFERENT STATIONS.

THE preceeding rule, however, must not be expected to give an accurate result, even in that particular temperature of the atmosphere, to which it is adapted, unless the specific gravity of the quicksilver, in the barometer, hath been the same at both stations, at the time of observation. If the specific gravity hath been different, at the different stations, the heights of the quicksilver, in the barometer, will *not* have been proportional to the densities of the air; that is, to the forces by which they have been sustained: and the calculation, built upon the supposition that they were so, becomes erroneous. The specific gravity of every material substance varies with its temperature. If the temperature of the quicksilver, therefore, hath been different, at the different stations, the difference of elevation, found by the foregoing rule, will require correction, though the mean temperature of the air may have been such as it prescribes⁽ⁱ⁾. No particular temperature of the quicksilver is necessary to the accuracy of the result of the preceeding rule, or to render the

(i) Observe, that the temperature of the quicksilver, in the portable barometer, will not be necessarily the same with that of the circumambient air, at the place and time of observation; there will generally be a considerable difference.

correction, I am now considering, 0; but only, that its temperature, whatever it be, be the same at both stations. If the *temperature* hath been the same, the *specific gravity* hath been the same; and if the *specific gravity* hath been the same, the *length* of the columns of quicksilver have been as the forces by which they were sustained, whatever the common specific gravity may have been. Columns of water, sustained in evacuated tubes, of sufficient length, would be proportional to columns of quicksilver in the barometer, at the same times and places, provided the temperature of the two water columns were the same, and that of the two mercurial columns the same; and, consequently, the difference of the logarithms of the water columns would be precisely the same as of the mercurial columns. For the water columns, and the mercurial columns, are only expressions of the same absolute magnitudes, the forces by which they are both sustained, in parts of different scales. But if the temperature of the quicksilver, or of the water, be different at the same time, at different stations; then, though we compare the water columns with each other, and the mercurial columns with each other, still we compare different things, though we call them by the same name. We compare fluids of different specific gravities; that is, we measure the pressure of the air at one place, in parts of one scale, and at the other place, in parts of another. The error is of the same kind, as if one should attempt to determine the proportion between two parts of a building, by measuring one with a Paris foot-rule, and the other with an English foot-rule, without attending

attending to the difference between Paris feet and London feet; and the same method must be taken to get rid of the error in both cases: we must ascertain the difference of the scales we have applied, and make allowance for it: we must know the length of the one in parts of the other; that is, in the particular case in question, we must be able to determine, from the observed height of the barometer, when the quicksilver is of any given temperature, what its height would have been, at the same place, and at the same time, if its temperature had been any other, that may be assigned. I have thought it necessary to be thus minute, in explaining the principles, upon which the correction in question depends; because it is a point, which is likely to be misunderstood, though in itself of no great difficulty. M. DE LUC himself hath fallen into a mistake, not with respect to the quantity of the error, but the manner of allowing for it; which, however, is of no other bad consequence, than that of lengthening the calculation unnecessarily. He imagines that a particular temperature of the quicksilver is necessary, that the error should be nothing; and to this he always reduces the observed heights of the quicksilver, in the barometer, at both stations ^(k). But the result of the computation would, in all cases, have been the same, if they had been reduced to any other given temperature; and therefore, it is always sufficient, to reduce the one to the temperature of the other. This little oversight I should have touched upon with more

(k) *Recherch. sur les Modif. de l'Atm.* §. 369—374.
reluctance,

reluctance, were it not of too little importance to derogate, in the least degree, from the general merit of M. DE LUC's elaborate and invaluable work.

The quantity of the correction in question, is thus determined. At times, when the barometer, in a temperate air, stood at 27 Paris inches, M. DE LUC tried what change was effected in its height, by changes of temperature, induced by art in the quicksilver, without any alteration in the state of the atmosphere; and, by repeated experiments of this kind, it was found, that the difference between the length of the column of quicksilver, successively heated to the temperature of boiling water, and cooled to that of melting ice, amounted to half a Paris inch exactly; that is, to $\frac{1}{54}$ th of the mean height ⁽¹⁾. But the whole extent of the thermometer's scale, from the temperature of melting ice to M. DE LUC's boiling point, being 178° of BIRD's Fahrenheit, the change of the height of the barometer, due to 1° of BIRD's Fahrenheit is always $\frac{1}{54 \times 178} = \frac{1}{9612}$. And if n denote the number of degrees of BIRD's Fahrenheit, in the difference of the temperatures of the quicksilver, in the barometers, at different stations, the reduction will be $\frac{n}{9612}$; that is to say, the height of the warmer column must be shortened by this part of its own length, or that of the cooler augmented by the like part, to reduce either, to the height it would have stood at, at the time and place of observation, with the temperature of the other.

(1) Recherch. sur les Modif. de l'Atm. §. 362—364.

These determinations of the effect of heat upon the column of quicksilver, in the Torricellian tube, shew the proportional alterations of the density of that fluid, by given increments, or diminutions, of heat. For the perpendicular height of a column, of any inelastic fluid, sustained in the Torricellian tube, by a given compressive force, must, by the known laws of hydrostatics, be as the densities of the fluid inversely. And as this proportion must obtain, whatever be the size, or figure, of the tube, there seems to be no method, by which the change of density, or the proportional expansion of quicksilver by heat, can be measured with more precision. These conclusions, therefore, may be of use in many physical enquiries; and there are many cases, in which it may be necessary, to reduce the observed height of the barometer, in one temperature, to another. Thus when that height is to be made the measure of the variable pressure, or of the density of the air, in some particular place, it will be necessary to choose some *standard* temperature, to which the observed length of the column may always be reduced. And it was this consideration, as I have gathered from many conversations with him, which gave occasion to M. DE LUC's mistake. He had settled it with himself, at his very first entrance upon these researches, that the point he was to keep constantly in view, as the ultimate object of his whole pursuit, was to find, in the variable length of the mercurial column, a measure of the pressure and density of the air. This, he saw, was only to be looked for in quicksilver of some

one constant temperature; and thus he became possessed with a general notion, of the necessity of reducing the observations of the barometer, to some standard temperature, upon all occasions.

But, for the particular purpose of computing differences of elevation, instead of attending to the correction of the observed heights of the quicksilver at all, it will be a readier way, to make the corresponding correction immediately, upon the difference of the tabular logarithms. If any quantity be diminished or increased, in a given proportion, the logarithm of its proportion to any other given quantity (that is, the difference of the tabular logarithms) is diminished or increased by a given magnitude; namely, by the logarithm of the given *ratio*, in which the variable quantity is altered. The logarithm of the *ratio* of x to $x + \frac{1}{9612}x$, or of 9612, to 9613, is 452^(m). Therefore, if n , as before, expresses the difference of temperature in degrees of BIRD's Fahrenheit, $\frac{n}{9612} \times 452$ is the correction to be applied to the difference of the logarithms of the observed heights of the quicksilver in the barometer. This correction is to be subtracted, if the temperature, at the upper station, hath been the cooler of the two, and to be added, in the contrary case. I am indebted to the ASTRONOMER ROYAL for the first hint of this elegant method of applying the correction in question.

(m) The reader will here recollect, that I speak of the tabular logarithms in the language of M. DE LUC. The logarithm of the *ratio* of 9612 to 9613 is, in the more common stile, 0.0000452.

The third of the annexed tables, shews the quantity of this correction corresponding to every value of π from 1° to 70° . The third column, to the right, exhibits the value of it in English fathom, in that particular temperature of the air, in which the difference of the tabular logarithms, of the heights of the quicksilver in the barometer, gives the difference of elevation in 1000ths of an English fathom. What that temperature is, will be determined in the sequel. It is to be observed, that the value of these corrections, in the fathom of any other country, will be the same, in that particular temperature, in which the tabular system measures the difference of elevations in 1000ths of the fathom of that country.

The rule, which I have given for applying this correction, supposes that it is previously known, which of the two stations is the highest; otherwise it will be doubtful, whether it should be added or subtracted. This doubtful case may actually happen in levelling a considerable tract of country, that is pretty even. The table, however, will still give the quantity of the correction. Add that quantity to the logarithm of the height of the cooler barometer; and the difference between this logarithm, so augmented, and the logarithm of the height of the other, is the difference of the logarithms of the observed heights, duly corrected: and the station of the cooler barometer, was the lower of the two, if its augmented logarithm exceeds that of the other: in the contrary case, the higher.

It hath been already observed, that the change of the density of quicksilver, by an alteration of its heat, must always be proportional to the increase

or contraction of the length of the column in the barometer; or, in other words, that the alteration, in the perpendicular height of the Torricellian column, is proportional to the change of the *solid content* of space, occupied by a given quantity of the fluid. Those who recollect the experiments of the celebrated BOERHAAVE, for measuring this change of volume, cannot but be struck with the great and singular disagreement between his conclusions and those of M. DE LUC. BOERHAAVE makes the whole expansion, produced by a change of temperature from the 0 of Fahrenheit, (or the forced congelation of *sal ammoniac*) to the heat of boiling water, $\frac{1}{52\frac{2}{3}}$ of the whole volume, which is very little more than M. DE LUC found to be produced by a change from 32° of Fahrenheit (or the dissolution of natural frost) to the heat of boiling water⁽ⁿ⁾.

(n) Vid. *Elementa Chæmiæ*, vol. i. p. 174. In this place, the author expressly treats of the expansion of quicksilver by heat; and the proportion, which he here assigns, hath been adopted by the writers of elementary physics. But, p. 165, having described an artificial cold, in which the thermometer had been sunk to -40° of Fahrenheit, he says, “*Novimus hoc argenti-vivi corpus ab illo gradu 40 infra 0; ad gradum 600 quo incipit ebullire contractum fuisse per partes 640 totius molis 10782.*” And, in the explanation of one of his copper plates, he describes a thermometer in which the quicksilver, in the greatest natural cold, just filled the bulb, which contained, he says, 11520 such parts as the tube contained 96. It seems probable, that the number 11124, which is given, p. 174, as the whole volume of the quicksilver in the temperature 0, (a degree of Fahrenheit being the unit) hath been deduced from the mean of a variety of experiments upon different instruments, for it differs not greatly from the mean of the two numbers 10782 and 11520.

Upon this it is to be observed, that BOERHAAVE's method gave him the expansion of his quicksilver, diminished by the whole expansion nearly of the vessel which contained it. DE LUC's gave him the expansion of his quicksilver, diminished by the longitudinal expansion of the scale applied to the barometer, by which the height of the column was measured. The vessel which contained BOERHAAVE's quicksilver was *glass*. The scale of DE LUC's barometer was *deal*. Glass is considerably altered in its dimensions by heat; deal but very little, in the direction of its longitudinal fibres: therefore, though M. DE LUC neglected the expansion of his scale, the expansion which he observed of the quicksilver would be nearly the whole; but BOERHAAVE must have reckoned it much too small, neglecting, as he says he did, the expansion of his glass^(o); and the difference is no more than what the expansion of the glass will very nearly reconcile. BOERHAAVE's expansion is too small, to be in proportion to M. DE LUC's, by about $\frac{1}{288}$ of the whole volume, or somewhat more than $\frac{1}{300}$. The expansion of glass, in length, by 180° , is, as I am informed, about $\frac{1}{1000}$ of an inch upon a foot, or $\frac{1}{1200}$ part; therefore, its expansion in solid content, or volume, will be very nearly $\frac{1}{400}$, by 180° ; and, by 212° , about $\frac{1}{331}$, or less than $\frac{1}{300}$.

(o) *Seposita ergo vitri interea dilatatione, &c.* These words may seem ambiguous; but the meaning seems sufficiently determined by what is said, p. 173. l. 2—5.

SECTION FOURTH.

Of the EQUATION for the TEMPERATURE of the AIR.

IF the temperature of the atmosphere hath been any other than is expressed by $+ 16\frac{3}{4}$ of M. DE LUC's scale, the result of the calculation, formed upon the preceeding rules, requires a further correction. This correction arises from a *variation of the length of the subtangent of the atmospherical logarithmic*; which, as hath been already remarked, is found to be not constantly the same. That this matter may be the better understood, it will be proper to state, in this place, what this subtangent is, in the nature of things, and upon what physical circumstances its length depends.

Imagine then, that instead of the atmosphere, in its natural state, the earth were surrounded with an inelastic fluid, of an uniform density throughout, equal to that of the natural atmosphere, in its lowest parts, which are contiguous to the surface of the earth; imagine also, that every atom of this homogenous fluid were urged towards the earth's center, with an accelerative force, equal to that of gravity, at the surface of the earth. Now the pressure of the atmosphere upon the earth's surface, or on any given part thereof, being, at all times, a finite, though not a constant force, of which the phænomena of the Torricellian tube are a sufficient proof, it is evident that it might, at any time, be equalled by the pressure of a finite quantity of this imaginary fluid; and that, to render the pressure of this fictitious atmosphere, upon the whole, or any part of the earth's surface, equal to that of the real atmosphere upon the like part, it would be requisite to assign to it some finite thickness or depth. *Now that*

that particular thickness or depth, of the imaginary fluid, which is, at any time, necessary to render an entire column of it a counterpoise for an entire column of the natural air, is, at that time, the length of the subtangent of the atmospherical logarithmic. In short, the subtangent of the atmospherical logistic, is the length of a column of such a fluid as I have supposed, which would be sustained in the Torricellian tube, by the pressure of the air, at the level of the sea, if we could suppose a tube of a sufficient length.

This is demonstrated by Mr. COTES, Harmon. Mens. p. 18. and by no one else, that I know of, with equal simplicity.

It is a manifest consequence from this, that the subtangent must always be as the pressure of the whole cylindrical column (upon a given part of the earth's surface) directly, and the density of the air, at the surface, inversely; and it may therefore appear to be repugnant to the theory already established, to suppose it subject to variation. For if the pressure be always as the density, upon which hypothesis the whole theory is founded, that which is always as the pressure directly, and the density inversely, can be no other than a constant quantity. But M. DE LUC's experiments prove, beyond a doubt, that the atmospherical subtangent is variable; therefore the density of the atmosphere, at the surface of the earth, at different times, is not proportional to the whole pressure, at such times, respectively. And in this there is nothing inconsistent with the foregoing theory, rightly understood. When the density of the air is said to be as the compressing force, this is to be understood of air in the

same state of elasticity; that is, in which the absolute force of elasticity, under all the different degrees of pressure, is constantly the same. (The absolute force of elasticity is measured by the force exerted between two particles of given magnitudes and figures, at a given distance.) But there is no reason, from any experiments, to conclude, that the density of the air will be simply as the compressing force, in different absolute forces of elasticity. On the contrary, since the proportion of the compressing forces is found to obtain, between the densities, when the absolute elasticity is constant; this alone is a proof, that the like proportion will not obtain, if the absolute elasticity be changed as well as the compressive force.

Imagine two parcels, similarly shaped, A and B, of different fluids, the same in all their other properties, and similarly, but unequally, elastic; that is, imagine the integrant particles of A to be equal in quantity of matter and bulk, and similar in figure, to the integrant particles of B, severally; and if each of the integrant particles of A, be itself an aggregate of lesser integrants, each of the integrant particles of B is to be understood to be a like aggregate of equal and similar lesser integrants, similarly composed; and whatever forces, except that of elasticity, are exerted between the particles of A, imagine equal forces, of the same kinds, exerted according to the same laws, between the particles of B. Likewise, imagine the equal and similar particles to be similarly situated, in their respective masses. Such fluids are the same in all respects, elasticity excepted. Further, imagine an elastic force to be exerted between every two integrant particles

particles of A, and to vary, with the distance, as any power or function of the distance whatever. Imagine an elastic force, varying according to the same power or function of the distance, exerted between every two particles of B. Thus the fluids are *similarly* elastic; but they are to be supposed *unequally* so; that is, the force exerted between two particles of the one, at any given distance, is to be greater than the force between two particles of the other, at the same distance. Now, if the law by which the elastic force varies, at different distances, in one of these fluids, as A, be such, that the densities of A, under different compressions, are to each other, as the compressive forces; the densities of B, compared with each other, under different compressions, will likewise be as the compressive forces. But the proportion of the compressive forces will not subsist between the densities of the two *different* fluids. The densities will not be the same, when the compressive forces upon A and B are equal; nor will they be *as* the compressive forces, when those forces are unequal. The densities will be as the compressive forces directly, and the absolute elasticities inversely; that is, the compressive forces will be as the densities and absolute elasticities jointly, which is easily demonstrated.

The compressive force upon any fluid, is to be estimated by the quantity of the pressure, acting perpendicularly upon a given surface, or plane section of it. Whatever degree of pressure acts upon the mass of an elastic fluid, its particles will approach or recede, till the distances of every two adjacent particles are such, that the whole elastic force, exerted perpendicularly against every surface or plane
section

section of it, is precisely equal to the pressure acting perpendicularly upon the same surface, in an opposite direction. And in the particular degree of density, determined by that distance of the adjacent particles, the fluid will remain, while the same degree of pressure is continued. Imagine therefore the densities of the two masses, A and B, to be the same; then, from the supposed similarity of the fluids, it follows, that the number of particles, exerting their elasticities upon any equal and similar sections of A and B, must be equal; and that the distances of the corresponding particles, from each other, in the two masses, must be the same: consequently the whole elasticities, exerted upon equal and similar sections of the two masses, will be as the absolute elasticities of which they are composed. Therefore the compressive forces are, in this case, as the absolute elasticities. Call the common density of the fluids D; the compressive force upon A, P; upon B, Π ; the absolute elasticities a , b , respectively. Let d denote some other density of the mass A, and p the compressive force corresponding to that density.

Now D and d are different densities of the same fluid A, under different compressive forces P, p .

Therefore, $p : P = d : D$ (by hypoth.)

But $P : \Pi = a : b$. (as hath been proved.)

Therefore, $p : \Pi = d \times a : D \times b$; that is, the compressive forces upon the different fluids A and B, when the densities are unequal, are as the densities and absolute elasticities jointly. Q. E. D. This demonstration is independent of any more particular hypothesis, concerning the law of the elasticity, than barely that it is the same in both masses; and such, in both, as to make the densities of either always proportional to

to the forces by which it is compressed. But to this condition, a particular law of elasticity is requisite; namely, that the force exerted between every two adjacent particles diminish as the distance between them is increased; and, adopting this law, the compressive forces might be proved to be in the proportion assigned, that of the densities and absolute elasticities jointly, by the same kind of reasoning as is used to demonstrate the 23^d proposition of the second book of the *Principia*, which is only the most simple case of the more general theorem now proposed.

Hence it follows, that if any causes act upon the atmosphere, to change the degree of its absolute elasticity, provided they act in such manner, as to change it equally at all heights; so that, though different at different times, it shall always, at any one time, be the same at all different heights; then, the densities in all different parts of the cylindrical column, resting perpendicularly upon a given small part of the earth's surface, will, at any one time, be as the compressive forces upon each part respectively. But the densities, in any one part of this column, at different times, when the absolute elasticities are different, will not be as the compressive forces upon that part, at such *different* times respectively. Hence the relation between the decrement of density, and the increment of height, reduced according to the construction of section second, will, at every time, be represented, by the decrement of the ordinates, and the increment of the asymptote of *some* logarithmic curve. But of *different* curves, at different times, or in different states of the absolute elasticity; that is, the length of the sub-

tangent of the curve, described in section second, will vary: for the difference of the subtangents is the only thing that constitutes a real difference in two logarithmic curves. If such curves have equal subtangents, though they have unequal ordinates, they are only *different parts* of the *same* curve.

The subtangent of the atmospherical logistic must always be as the pressure of the whole column of the atmosphere directly, and the density, at the earth's surface, inversely; therefore it is directly as the absolute elastic force. For call the subtangent S ; the pressure at earth's surface, P ; the density, D ; the absolute elasticity, A .

Now that S is as $\frac{P}{D}$ is obvious (from p. 245.); but $\frac{P}{D}$ is as A (by what hath now been proved). Therefore S is as A .

Now *heat* ^(p) is one cause, which is well known to influence the absolute elastic force. An increase of heat increases elasticity; and elasticity is diminished by a diminution of heat. Accordingly M. DE LUC's experiments shew, that when the temperature of the air is uniform (or the same at all heights) the subtangent of the atmospherical curve is increased, or diminished, exactly in proportion to the increment or decrement of the uniform temperature, as indicated by the mercurial thermometer. His conclusion, from repeated experiments on the mountains near Geneva, is, *that, if* L *denote the difference of the tabular logarithms of the heights of the quicksilver, at two stations, corrected for the difference of the temperature of the quick-*

(p) BOERHAAVE, *Elementa Chimiæ*, vol. i. p. 456, &c.
silver,

filver, and n denote the difference of the temperature of the air, in degrees of his own scale, above or below $+ 16\frac{1}{2}$, then $L \pm \frac{n}{215} L$, is the difference of elevation in 1000ths of a Paris toise. The correction $\frac{n}{215} L$, to be added or subtracted, according as the temperature of the air is above or below $16\frac{1}{2}$; or according as n is a difference in excess or defect (9).

But in any given value of n , the proportion of $\frac{n}{215} L$ to L is given, being that of n to 215. There-

fore, $L \pm \frac{n}{215} L$ is the logarithm of the ratio of the heights of the barometer, in a system of logarithms, in which the subtangent of the curve is to so many 1000ths of a Paris toise, as are expressed by the subtangent of the Briggian system, as $215 \pm n$ to 215. And this being the case, whatever be the value of L , in every given value of n , it is evident, that the variation of the *modulus* of the atmospherical system, or of the subtangent of the atmospherical curve, is always as n , the variation of temperature.

But a uniform temperature is a condition of the atmosphere, which rarely obtains, within small distances, at least, above the earth's surface; therefore the more usual case of unequal temperatures must be considered.

When different temperatures obtain, at different heights, at the same time, they must render the absolute elasticities, at those heights, unequal. Thus the hypothesis of M. BOUGUER will take place;

(9) Recherch. sur les Modif. de l'Atm. §. 607—611.

who, from a great number of experiments, made upon the CORDILEROS, and at various other heights above the level of the sea, concluded, that the absolute elasticity of the air, which he calls, “the intensity of the elastic force,” must be different at different elevations. His conjectures indeed about the cause from whence the difference might arise, are not the most natural; and in one point he was evidently mistaken: namely, that he imagined the absolute elasticity to be constant at every *given* elevation; and accordingly he hath traced the curve, which, according to his experiments, exhibits the laws of its variations from one height to another. This, it must be confessed, was trusting his experiments too far, which were not made with the most accurate instruments; and this may have given occasion to his learned countryman M. DE LA LANDE, to suppose that this curve may rather exhibit the deviations of his experiments from the truth, than any thing which really obtains in nature^(r). His general hypothesis, however, of a difference of absolute elasticity, in different parts of the atmosphere, *must* obtain, whenever the temperatures of such different parts are unequal; unless the effect of the inequality of temperature were to be compensated by the synchronous unequal operations of some other cause. That it does in fact obtain, when the temperature is unequal, is proved, beyond a doubt, instead of being refuted, by M. DE LUC’s experiments; and this is to be considered, as a further instance of their exact agreement with the genuine conclusions of accurate theory.

(r) Connoissance pour l’année 1765, p. 215.

When

When the absolute elasticities are different at different heights, the densities will no longer be proportional to the pressures; and the change of density, throughout the whole column of the atmosphere, will no longer be represented by any *one* logarithmic. But to small distances on one side or the other of different heights, it may still be nearly represented by parts of *different* logarithmics. Now this is really the case, according to M. DE LUC: for, when the temperature, at two different heights, hath been different, he finds, *that the difference of those heights will still be expressed in 1000ths of a Paris toise, by the preceeding formula, viz.*

$L \pm \frac{n}{215} L$, if n be understood to denote the difference between the constant temperature $+ 16\frac{1}{2}$, and that which is the mean of the different temperatures of the two places of observation ^(s); that is to say, though the temperature of that portion of the column of the atmosphere, which is intercepted between the level of the two places of observation, hath not been the same, perhaps, in any two different parts; yet the variation of density and height will be exhibited, through the whole of this small space, without sensible error, by the curve, which would have represented them strictly, if the entire column had been of the mean temperature of the extremes of this portion of it; namely, by the logarithmic, whose subtangent is expressed in 1000ths of a Paris toise, by $B \pm \frac{n}{215} B$, B denoting the subtangent of the Briggian system. Imagine, therefore, that the barometer and thermometer have been observed at

(s) Recherch. sur les Modif. de l'Atm. §. 663.

the same time, at three different heights. Put the difference of the tabular logarithms of the heights of the quicksilver, at the 1st and 2d elevation L ; at the 2d and 3d, L'' . Imagine the thermometer at the 1st, or lowest height, to have been $+ 16\frac{3}{4}$; at the 2d, $16\frac{3}{4} + 2n$; at the third, $16\frac{3}{4} - 2m$.

Then the diff. between 1st and 2d elevation, $= L + \frac{n}{215} L$

between 2d and 3d, $= L'' + \frac{n-m}{215} L''$

that is, the variations of height and density between each two of these three places, are nearly exhibited by two different logarithmics, the subtangents of which are

$B + \frac{n}{215} B$; $B + \frac{n-m}{215} B$; therefore, the variations

of height and density, throughout the atmosphere, are not represented by any *one* logistic, when the temperature is unequal (as they would be, if, notwithstanding an inequality of temperature, the absolute elasticities were the same in all parts); but by parts of different logistics, at different heights; as they should be when the absolute elasticities are different, in different parts, and can only be considered as uniform to a small distance, above or below any height assigned.

The mention of M. BOUGUER occurred so naturally upon this occasion, that I must have reproached myself with an impiety towards the ashes of a man, whose memory will ever be dear to science, had I not attempted to vindicate his conjectures upon the point under immediate consideration, from an animadversion thrown out, in very general terms, with little tenderness, and only just in part. At the

the same time, I have pointed out, in what his principal mistake consists; which was, as M. DE LUC well observes, that, instead of seeking a rule which should be general for all heights, and vary at every height, in proportion to the temperature, he set himself to find one which should be general for all temperatures, at a particular elevation. The consequence likewise, which he would have deduced from the unequal elasticity of the particles of the air, that the least elastic would be driven to the bottom, was undoubtedly erroneous. The excess of temperature may fall sometimes in the lower parts of the atmosphere, and sometimes at greater heights; and where the greater temperature is, there, *ceteris paribus*, the elasticity will be greater. The argument, which M. DE LUC directs against the existence of sensible inequalities of elastic force in aggregate masses of the atmosphere, derived from the supposed effect of the winds, throughout all the regions exposed to their agitations⁽¹⁾, militates only against the probability of *permanent* inequalities in *given* places, arising from supposed specific differences in the original constitution of the particles of the air; not against such temporary inequalities, as we ascribe to the occasional energy of extraneous causes. An inequality of temperature undoubtedly exists, in aggregate masses, more frequently than the opposite. And from an inequality of temperature, whether in the aggregate or the discrete, necessarily follows, for the time, an analogous inequality of absolute elastic force.

(1) Recherch. sur les Modif. de l'Atm. §. 328.

Having now sufficiently explained, what the correction is, for a variation of the temperature of the air, and whence it arises, I proceed to reduce M. DE LUC's *formula* to BIRD's Fahrenheit, and a scale of English fathom.

SECTION FIFTH.

M. DE LUC's RULES reduced to ENGLISH SCALES.

THE whole of this reduction I divide into three problems.

PROBLEM FIRST.

To find the length of the subtangent of the atmospheric curve, in thousandths of a Paris toise, the mean temperature of the air being given in degrees of BIRD's Fahrenheit.

BY the comparison of M. DE LUC's scale with BIRD's Fahrenheit, it appears, that $+16\frac{1}{2}$ of the former corresponds to $+69,25$ of the latter. Hence $69,25$ is the temperature in BIRD's Fahrenheit, in which, the subtangent, of the atmospheric curve, is equal to so many 1000ths of a Paris toise, as are expressed by B, the subtangent of the Briggian system. But the atmospheric subtangent is increased or diminished by $\frac{1}{215}$ of this quantity for every degree of M. DE LUC's scale

scale, above or below this given temperature; and a degree of M. DE LUC's scale is to a degree of BIRD's Fahrenheit as 178 to 80. Therefore, the subtangent varies by $\frac{8}{3827}$ of the same quantity, for every degree of BIRD's Fahrenheit, above or below the given temperature. Hence, if n denote the difference of the temperature of the air, or, in the case of unequal temperatures, the difference of the mean of the temperatures of the two stations, above or below 69,25 in degrees of BIRD's Fahrenheit; then, $B \pm \frac{n \times 8}{3827}$ B is the length of the subtangent in 1000ths of a Paris toise: that is, the subtangent of the atmospherical curve, in the temperature $69,25 \pm n$, is to so many 1000ths of a Paris toise, as are expressed by the modulus of the Briggian system, as $3827 \pm \frac{n \times 8}{3827}$ to 3827.

PROBLEM SECOND.

To determine the temperature, in which the length of the subtangent of the atmospherical curve is expressed in thousandths of an English fathom, by the subtangent of the Briggian system.

FROM the number 69,25 subtract the 8th part of the number, to which 3827 bears the proportion of 10000 to 617; that is, from 69,25 subtract 29,51; the remainder 39,74 expresses the required temperature, in degrees of BIRD's Fahrenheit.

For, let S , Σ represent the subtangents of the atmospherical curves in the temperatures 69,25 and 69,25 — 29,51, respectively.

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Then $S : \Sigma = 3827 : 3827 - \overline{29,51 \times 8}$ (by prob. 1.)

But $3827 : 3827 - \overline{29,51 \times 8} = 10000 : 10000 - 617 = 10000 : 9383$.

Therefore, $S : \Sigma = 10000 : 9383$; that is, as one Paris

toise to one English fathom. Therefore, whatever multiple S is of the Paris toise, or any part thereof, the same multiple is Σ of the English fathom, or its like part. And the length of S is expressed in 1000ths of a Paris

$S : \Sigma$ toise, by the number which is the *modulus* of the Briggian system (by prob. 1.); therefore, the length of Σ is expressed by the same number, in 1000ths of an English fathom.

PROBLEM THIRD.

To find the equation for every degree of BIRD'S Fahrenheit in the mean temperature of the air, above or below 39,74.

CALL the variation of the length of the subtangent, corresponding to an increment or decrement of one degree of BIRD'S Fahrenheit, V; and let S, Σ , as before, represent the subtangents corresponding to the temperatures 69,25 and 39,74.

Now $V : S = 8 : 3827$ (by prob. 1.)

And $S : \Sigma = (3827 : 3827 - \overline{29,51 \times 8}) = 3827 : 3591$ very nearly (by prob. 1.)

Therefore, $V : \Sigma = 8 : 3591$.

that is, $V = \frac{8}{3591} \Sigma$. ($= \frac{1}{444} \Sigma$ nearly).

Hence the length of the subtangent of the atmospheric curve, in any temperature, $39,74 \pm n$ is to its length in the temperature 39,74 as $3591 \pm 8 \times n$ to 3591; that is, putting B for the subtangent of

the Briggian curve, $B \pm \frac{n \times 8}{3591} B$ is the length of the subtangent of the atmospheric curve, in 1000ths of an English fathom. And putting L for the difference of the tabular logarithms of the observed heights of the barometer, at two stations, corrected by the equation for the temperature of the quicksilver, and n for the mean difference of the temperatures of the air, in degrees of BIRD'S Fahrenheit, above or below 39,74, the difference of the elevation of the two stations, is expressed in 1000ths of an English fathom by $L \pm \frac{n \times 8}{3591} L$.

Upon these principles I have made a table, by which the equation $\frac{n \times 8}{3591} L$ may be computed for any temperature not above + 80, nor below 0 of BIRD'S Fahrenheit, and for any height less than 10000 fathom.

It hath already been remarked, that the temperature of the quicksilver, in the portable barometer, may happen to be very different from that of the air, at the place and time of observation. For this reason, M. DE LUC advises, that every portable barometer should be furnished with two thermometers; one fixed to the frame of the barometer, to indicate the temperature of the quicksilver; another, to be exposed, at the time of observation, to the open air. The use of the annexed tables might be answered by a particular division of the scales of these thermometers. The thermometers which accompany the two barometers, made for the society, by Mr. EDWARD NAIRNE, under my directions,

are BIRD's Fahrenheits ; but, to the fixed thermometer, I have had a particular scale applied, in which the fundamental interval, between melting ice and BIRD's boiling point, is divided into 81 equal parts. The equation for the temperature of the quicksilver is one fathom for every degree, of this scale, in the difference of temperatures. The thermometers, for the temperature of the air, have each a scale, in which the fundamental interval between melting ice and BIRD's boiling point, is divided into 120 equal parts. The point of 0 is placed at the 5th of these divisions above melting ice. If n be the mean height of the thermometer, in degrees of this scale, at the two stations, the equation for the temperature of the air is $\frac{n}{300}$ L. + or —, according as n is positive or negative.

The place of the point of 0 upon the former scale is indifferent. It was put very low, that the temperatures of the quicksilver, at both stations, might always be above it. The computation, however, is rendered easier by the tables now given, than it can be by any such peculiar divisions of the scale.

It is to be particularly observed, that M. DE LUC always exposed the thermometer, by which he measured the temperature of the air, *to the sun*, if it happened to shine ; but then the ball of his thermometer was always quite detached from the frame, which is a necessary precaution in this manner of using it. If the temperature of the air were measured by a thermometer, exposed in the shade, I cannot but think the quantity of the equation would be different.

His

His reason for not shading his thermometer from the sun, was this: when one side of the thermometer is exposed to the sun's rays, the other is in its own shade; therefore, in this situation, he thought it the just measure of the *mean* temperature of the air, which is neither the temperature of that part on which the sun's rays fall, nor of that from which they are intercepted, but less than the one, and greater than the other, or a mean between the two^(u). I confess, I should have expected an irregularity, from heat excited by the rays of light, in their passage from the air into the glass, and from the glass into the quicksilver; and should therefore have exposed my thermometer in the shade; but the success of M. DE LUC's experiments seems sufficiently to justify the method he hath taken.

I shall close this section with a brief solution of two physical problems, for which this seems the proper place.

PROBLEM FOURTH.

To compare the densities of the air, at any given elevation above the surface of the earth, in different temperatures.

IF the barometer hath been observed, at any given elevation above the earth's surface, in different temperatures, reduce the observed heights of

(u). Recherch. sur les Modif. de l'Atm. §. 533—536.

the quicksilver to a common temperature, if the temperatures of the quicksilver, as well as the air, have been different, at the different times of observation. Find the subtangent for each temperature of the air (by problem 3d); divide the corrected heights of the quicksilver by the subtangents corresponding to the observed temperatures of the air. The quotients are as the densities in these temperatures respectively; that is, calling the heights of the mercury reduced, P, Π ; the temperatures, T, Θ ; the subtangents, S, Σ ; and the densities, D, Δ , $D : \Delta = \frac{P}{S} : \frac{\Pi}{\Sigma}$. For the densities are as the compressive forces directly, and the absolute elasticities inversely (by section 4th); and the absolute elasticities are as the subtangents (by section 4th); whence the truth of the rule is manifest.

The most convenient method, however, for practice will be, to make the density of the air, in some given state of the barometer and thermometer, a standard, with which to compare the densities in all other states of these instruments. Suppose, for instance, at the level of the sea, 30 inches be taken for the standard height of the barometer, and $+40$ for that of the thermometer. Let D be the density of the air, at the level of the sea, when the barometer is 30 inches, and the thermometer $+40$. Put $P = 30$ inches, and $S =$ the subtangent of the logarithmic corresponding to the temperature $+40$. Now in another temperature $40 + n$, let the subtangent be $S + \frac{b}{m} S$. And let the height of the barometer P be changed into $P + \frac{a}{r} P$; and let Δ be
the

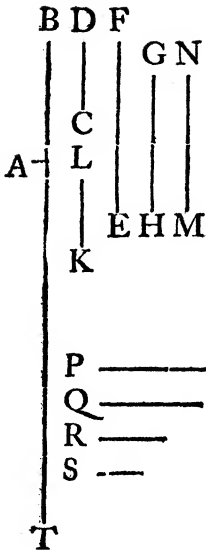
the density at the level of the sea, in the temperature $40 + n$, and height of the barometer, $P + \frac{a}{r} P$. Then the difference between Δ and D , or $\Delta - D$, will be $D \times \frac{ma - br}{r \times m + b}$. Where observe, that b is positive or negative, according as n is positive or negative.

By this problem, the densities of the air at a given height, in different temperatures, are to be compared with each other. Even good writers have hitherto been generally run away with by a notion, that the air's density, at a given elevation, at different times, would be always truly measured by the length of the mercurial column; than which nothing can be more erroneous, as hath been at large explained in the 4th section.

PROBLEM FIFTH.

The height of the quicksilver in the Torricellian tube, and the temperature of the air being given, at a given elevation, above the level of the sea, to compare the density of the air with that of the quicksilver, at the time and place of observation.

LET TA represent the semi-diameter of the earth; T being the center, and A at the surface. Let B be a place at a given height, AB, above the surface. Let CD be the observed length of the quicksilver in the Torricellian tube, at the place B, in a given temperature of the air. It is required to find



find the proportion between the density of the air, and that of the quicksilver, the observed temperature being supposed to obtain uniformly throughout the whole atmosphere; or at least, throughout a great portion of it, on all sides of B.

Suppose it done; and that the density of the air, at the time and place of observation, is to the density of the quicksilver in the barometer, as the right line CD to EF. Now the temperature of the air being given, the subtangent of the logarithmic corresponding to the given temperature is given (by prob. 3d). Let GH be equal to that subtangent; and let KL be the perpendicular height of the mercurial column supported in the barometer at A, the earth's surface, at the time of observation at B. Now since EF is to CD as the density of quicksilver, in the barometer, to the density of the air at B, and CD is the perpendicular height of the column of quicksilver sustained in the Torricellian tube at the place B, at the time of observation, it is evident, that EF is the perpendicular height of a column of an inelastic fluid, of the same density with the air at B, which, if it were urged with an accelerative force through its whole length every where equal to that of gravity at B, would, by its compressive force, sustain the mercurial column CD: for the accelerative forces

supposed

supposed to act upon the fluids in the columns EF, CD, being equal, and the heights of the columns being, reciprocally, as the densities of the fluids, the compressive forces of the columns will be equal. But GH is the perpendicular height of a column of an inelastic fluid, of the same density with the air at A, the level of the sea, which, if the whole were urged with an accelerative force equal to that of gravity at A, would sustain the mercurial column KL (by sect. 4). Suppose MN equal to the perpendicular height of a column of an inelastic fluid, of the same density with the air at B, which, if the whole were urged with an accelerative force, every where equal to that of gravity at A, would sustain the mercurial column CD. It is evident that EF is to MN as the accelerative force of gravity at A to the accelerative force of gravity at B; for EF and MN are columns of fluids, of the same density, exerting equal compressive forces. Therefore, the heights of the columns must be reciprocally as the accelerative forces by which the compressive force is produced. But gravity at A is to gravity at B as the square of TB to the square of TA. Therefore, $EF : MN = TB^2 : TA^2$. Again, GH and MN are columns of fluids of different densities, acted upon by equal accelerative forces; therefore, the compressive forces which they exert, that is, which they sustain, will be as their heights and the densities of the fluids jointly; that is, if P, Q be as the compressive forces of those fluids, and R, S as their densities, then $P : Q = GH \times R : MN \times S$. But the compressive forces sustained by these columns are the pressures of the air at A and B; also

the densities of the fluids are the densities of the air at A and B. Therefore the compressive forces are as the densities; that is, $P : Q = R : S$. Therefore $GH \times R : MN \times S = R : S$. Therefore GH and MN are equal. Therefore $EF : GH = EF : MN$; but it hath been shewn, that $EF : MN = TB^2 : TA^2$. Therefore $EF : GH = TB^2 : TA^2$. But AB being given, TB is given, and the proportion of the square of TB to the square of TA is given. Therefore the proportion of EF to GH is given. And GH, is given as hath been shewn. Therefore EF is given, and the proportion of the given line EF to the given line CD, or of the density of the quicksilver to the density of the air, at B, is given. Q. E. I.

COMPOSITION.

Find the subtangent GH competent to the given temperature; find a line, EF, to which that subtangent shall bear the duplicate proportion of TA to TB; that is, of the earth's semi-diameter to the earth's semi-diameter increased by the given elevation of the place of observation, above the level of the sea. As that line EF to the observed height of the quicksilver, so is the density of the quicksilver, in the barometer actually employed, to that of the air, at the time and place of observation.

Thus the specific gravity of air may be found, comparing it with quicksilver, and by means of quicksilver with other fluids. Though it is only the most simple case of this problem that can ever come into practice, I chose to discuss it in its most general extent;

tent ; as this is the only one method which gives entire satisfaction to the mind.

SECTION SIXTH.

I HAVE now gone through the exposition and reduction of M. DE LUC's rules, for measuring heights by the barometer. The consonance of them with theory appears, upon a strict examination, to be such, as strongly confirms the principles, upon which the theory is founded. I shall conclude these disquisitions, with pointing out some further objects of enquiry, concerning the modifications of the atmosphere, naturally arising out of M. DE LUC's discoveries, in conjunction with the theory already established.

1. *It is probable, that the absolute elasticity of the air may be affected by various causes besides heat.* The degree of humidity must occur to every one, as a circumstance, which may reasonably be suspected to have some influence upon it ; and, perhaps, the state and quantity of the electricity of the air may have more.

2. *If M. DE LUC's formulæ are to be admitted as universally true, in all imaginable temperatures, there is a given temperature in which the elasticity of the air would be destroyed, and, in any lower temperature, it would be negative ; that is, the repulsion would be changed into attraction.* This given temperature is, $-409,13$ of BIRD's Fahrenheit ; for if $B + \frac{\pi \times 8}{3591} B$, be the length of the
M m 2
subtangent

subtangent of the atmospherical curve, in 1000ths of an English fathom, whatever be the value of n , then, when $n = -448,875$, $B - \frac{8 \times n}{3591} B = 0$; that is, the subtangent vanishes; and the absolute elastic force, which is always as the subtangent (by §. 4.) must vanish with it; but when $n = -448,875$ the temperature is $-409,13$.

Perhaps it may be thought more probable, that the variation of the subtangent, or of the elastic force, is not precisely as the variation of temperature. If the subtangent changes in a *geometrical* proportion while the temperature, as shewn by the thermometer, changes arithmetically, the subtangent, or the absolute elastic force, will not vanish with any assignable decrement of temperature; and in that temperature, in which it should vanish, accords to M. DE LUC's *formulæ*, it will still remain more than $\frac{3}{10}$ ths of what it is in the temperature $+39,74$; and yet the equation, for an increment or decrement of temperature, amounting to 40° , will not differ from M. DE LUC's by more than four fathom in the height of 1000. I must repeat, that I am now only pointing out the conclusions of theory, as hints of further enquiry. I do not mean to substitute this hypothesis as more accurate than M. DE LUC's in practice; I do not affirm, that it is more true in theory. I mean only to suggest, that if M. DE LUC's *formulæ* are admitted as mathematically true, a consequence will follow, which may seem to some unlikely to obtain in nature. That however this consequence, if otherwise improbable (which is not the opinion to which, for
my

my own part, I incline) is not to be too hastily adopted, upon the bare evidence of its arising out of M. DE LUC's *formulae*; because the general extent of these *formulae* is more than experiment hath hitherto proved. And I shew, by a very simple instance, that it is easy to imagine other laws, between the temperature and the elasticity, which, to all practical purposes, would be the same as M. DE LUC's, within the utmost limits of his experiments, and even beyond them, and yet differ from it in remote consequences.

If M. DE LUC's *formulae* be universally and strictly true, the consequence I have pointed out must be admitted; and it will follow, that the force of elasticity owes its first production and origin to temperature: or, at least, that the relation between heat and elasticity, if not the most intimate one of cause and effect, is that which stands the next in the scale of natural affinities, namely, that they are inseparable effects of some common cause. And these conclusions will hold, if the form of the general expression be true, though the quantity of the equation $\frac{n \times 8}{3591}$ B should require a correction: for a change of the fraction $\frac{8}{3591}$ will only alter the particular temperature, in which the subtangent and elasticity vanish.

3. *The diminution of the air's density, as we ascend from the surface, is subject to a limit.* This limit is different, in different heights of the barometer, at the level of the sea, and even in a given height of the barometer, in various temperatures. But the density even at an infinite height is never 0, or infinitely small. In fig. 1. through

C draw

C draw CF, parallel to AD, meeting the atmospherical curve in F. CF is the density of the atmosphere, at an infinite height (by p. 231). The proportion of CF to the density at the level of the sea is given in any uniform given temperature of the air. For the uniform temperature being given, the subtangent of the curve DF is given; and consequently, the proportion is given, of which the given line CA is the logarithm. The diminution of compressive force is subject to a corresponding limit.

This curious circumstance hath been remarked by no one, that I know of, but Dr. BROOKE TAYLOR, whose writings are too little attended to, even among mathematicians, from an opinion which prevails of their obscurity. That consummate geometer seems indeed to have thought it improbable, that a finite density of the atmosphere, at infinite distances, though the necessary consequence of the theory, should actually exist; and, for this reason, he would imagine such a law of the elastic force, as should render the density of the atmosphere, beyond a certain height, much greater than in proportion to the compressive force; and circumscribe the whole within narrow limits^(w). And if the theory inferred a *great* density at infinite or even at great heights, such a density as would sensibly disturb the motions of the planets, it would be necessary to have recourse to some such hypothesis; but, as this is not the case, I see no such necessity. I know not, for what reason, mathematicians have been afraid to admit the infinitude of the atmosphere of the earth; whether they thought it would bear hard upon the Newtonian doctrine

(w) Method. Increment. prop. 26. Scholium.

of a void ; or, that it implied the infinitude of matter. But neither the one nor the other of these consequences is to be apprehended : for neither the phænomena of nature, nor the principles of the Newtonian philosophy, require, that there should be any where a great chasm in the universe ; or that the whole material world should be actually circumscribed within any finite space. A large proportion of *pore*, or interspersed vacuity, is sufficient for all purposes. Nor doth an absolute infinity of matter by any means follow from the hypothesis of an infinite *number* of finite masses ; and an infinite number of finite masses is all that is implied in the notion of a rare elastic fluid, diffused throughout infinite space. I agree, indeed, with Mr. COTES, that there are no *data* from which any great altitudes of the atmosphere can indubitably be concluded, in the way of experiment : but I do contend, that there are no *data*, from which the supposition of its infinite height can, in the same way, be disproved. And this may justly be held more probable than the contrary, as being the consequence of a theory which hath never yet, in any instance, proved fallacious ; and this I venture to assert, with the less hesitation, as, besides the evident reason of the thing, the great authority of NEWTON is on my side. The infinite extent of the earth's atmosphere is manifestly supposed, in that subtle disquisition concerning the tails of comets, which occurs in the posthumous work *De Systemate Mundi*. Instead of framing hypotheses, therefore, to remove imaginary difficulties, I shall pursue the theory as far as it will lead ; and my next step shall be, to exhibit the successive rarefactions of the atmosphere, as we ascend from the earth, in a table.

Heights

| | Heights in miles | Volume. |
|----------|----------------------|------------------------------------|
| 0,000 | 00,0 | 1 |
| 0,01 | 40,0 | $\sqrt[3]{3069}$ ² |
| 0,02 | 80,8 | $\sqrt[3]{3069}$ ³ |
| 0,03 | 122,6 | $\sqrt[3]{3069}$ ⁴ |
| 0,04 | 165,0 | $\sqrt[3]{3069}$ ⁵ |
| 0,05 | 208,5 | $\sqrt[3]{3069}$ ⁶ |
| 0,06 | 252,1 | $\sqrt[3]{3069}$ ⁷ |
| 0,07 | 298,2 | $\sqrt[3]{3069}$ ⁸ |
| 0,08 | 344,5 | $\sqrt[3]{3069}$ ⁹ |
| 0,09 | 391,9 | $\sqrt[3]{3069}$ ¹⁰ |
| 0,10 | 440,3 | $\sqrt[3]{3069}$ ¹¹ |
| 0,11 | 489,7 | $\sqrt[3]{3069}$ ¹² |
| 0,12 | 540,6 | $\sqrt[3]{3069}$ ²⁰ |
| 0,20 | 990,6 | $\sqrt[3]{3069}$ ³⁰ |
| 0,30 | 1698,1 | $\sqrt[3]{3069}$ ⁴⁰ |
| 0,40 | 2641,7 | $\sqrt[3]{3069}$ ⁵⁰ |
| | <i>Semidiameter.</i> | $\sqrt[3]{3069}$ ⁶⁰ |
| 0,50 | 1,00 | $\sqrt[3]{3069}$ ⁷⁰ |
| 0,60 | 1,50 | $\sqrt[3]{3069}$ ⁸⁰ |
| 0,70 | $2\frac{1}{3}$ | $\sqrt[3]{3069}$ ⁹⁰ |
| 0,80 | 4,00 | $\sqrt[3]{3069}$ ⁹⁸ |
| 0,90 | 9,00 | $\sqrt[3]{3069}$ ^{98,36+} |
| 0,98 | 49,00 | $\sqrt[3]{3069}$ ¹⁰⁰ |
| 0,9836 + | 62,00 | |
| 1,00 | Infinite. | |

This table shews the rarefactions of the atmosphere at different heights, above the surface of the earth, upon a supposition, that the temperature, throughout the whole, is that which is indicated by + 40° of BIRD'S Fahrenheit. The middle column shews the heights above the surface, corresponding (according to the construction of the atmospherical logarithmic, §. 2) to the decimal parts of the earth's semi-diameter expressed by the numbers in the first column; and in the third the corresponding rarefactions, or the proportions of the spaces which a given quantity of air would occupy at the different heights, are expressed by the powers of the number 3069.

The principle and method of the construction of this table is obvious. I compute the length of the mean semi-diameter of the earth in thousandths of an English fathoms. I divide that number by 100. The quotient is the tabular logarithm of the 100th root of a number, which is to unity, as the density at the surface to the density at an infinite height, in the

the temperature supposed; or, which is the same thing, as the space occupied by a given quantity of air, at an infinite height, to the space occupied by the same quantity, at the surface. This root I call the *radical number*. Then, imagining the earth's semi-diameter, divided into 100 equal parts, and numbering those parts 1, 2, 3, &c, downwards, towards the center, I compute the heights, above the surface, corresponding to those parts successively, according to the construction of the atmospherical logarithmic, §. 2; and, writing the resulting numbers in the second column, in the third, I write the powers of the radical number, increasing by unit, in regular succession downwards.

The proportion in which the atmosphere will be rarefied, at given heights above the surface, will be very different, in different temperatures. It may always be exhibited by a table of this form, but every different temperature will have its own radical number. The radical number, in the temperature $+40$, is 3069; and, for any other given temperature, may be thus found. Call the tabular logarithm of 3069, y ; and let n be the number of degrees of BIRD'S Fahrenheit, in the difference between the given temperature, for which the radical number is to be found, and $+40$. Then

$y \pm \frac{n}{449 \mp n} y$ will be the tabular logarithm of the radical number, for the temperature assigned: observing, that n is to be positive or negative in the denominator of the coefficient $\frac{n}{449 \pm n}$, according as

the given temperature is greater or less than $+40$; and that the term $\frac{n}{449 \pm n} y$ is negative when n in the denominator of its coefficient is positive, and *vice versa*. Thus the radical number, for the temperature 0, is $6731,2 +$.

This table is not intended for any practical purposes; but merely for speculative amusement. They, who take delight in the contemplation of final causes, will remark, with admiration, how large a part of the whole rarefaction of the atmosphere is performed on this side of the moon. Indeed there is comparatively but a very small part performed beyond it: so that the moon revolves at a distance where the resistance, from the earth's atmosphere, is reduced almost to its *minimum*. For if we imagine a series of quantities, consisting of 101 terms, decreasing from the first continually in geometrical proportion; and such, that the first and greatest shall be to the last and least, as the density of the atmosphere, at the earth's surface, to the density at an infinite distance; the density, at the mean distance of the moon, will be less than the 99th of these, or the least but two.

4. *An increase of temperature rarefies the lower regions of the atmosphere, more in proportion than the upper, and brings the constitution of the whole nearer to that of an uniform density.* Though the effect of heat alone, upon every orb of the atmosphere separately, is to rarefy each, perhaps, proportionally, yet its operation upon the different orbs is so modified, by the different degrees of pressure they

they sustain, and by the communication between them, that the greater the temperature of the atmosphere, though uniform throughout, the less will the proportion be of the density at any given height to the density at any greater given height. For the greater the temperature, the greater the subtangent; and the greater the subtangent, the less the proportion of AD to CF, of which the given line AC is the logarithm. (*Vid.* fig. 1.)

5. *If at any height above the surface of the earth a given alteration of temperature diminish the air's density in the same proportion, as it increases the absolute elasticity, or vice versa, the pressure of the superincumbent atmosphere, at that height, will remain unchanged. At all lower heights, the pressure will be less, than in a cooler condition of the atmosphere, and greater at all greater heights. On the contrary, the pressure at all lower heights will be greater than in a warmer condition, and at all greater heights less.*

For let CA (fig. 2.) represent the semi-diameter of the earth, the curve DEF the atmospherical logarithmic for a certain temperature, and GHK the logarithmic for another greater temperature. Let the ordinates of the two curves AD, AG be as the densities, at the earth's surface, in the different temperatures, to which the curves belong, respectively. Then it is evident, the ordinates βE , βH , drawn through any other point β in the asymptote, will be as the densities, at the height to which $A\beta$ corresponds, in the different temperatures, respectively. Now, suppose that the density of the air, at the height B, in the greater temperature, is less than the density at

the same height in the cooler temperature, in the same proportion, as the absolute elasticity in the greater temperature exceeds the absolute elasticity in the less. Then I say first, that the pressure of the superincumbent atmosphere at B, is the same in both temperatures. For take $C\beta = \frac{CA^2}{CB}$, and draw the ordinate βE , cutting the curves in E and H, and through E and H draw tangents to the curves, EL, HM, meeting the common asymptote, AC, in L and M. Now the subtangents $\beta L, \beta M$, are as the absolute elasticities in the different temperatures (by p. 250.). And $\beta E, \beta H$ are as the densities at B (by construction). Therefore $\beta H : \beta E = \beta L : \beta M$. Therefore $\beta H \times \beta M = \beta E \times \beta L$. But the rectangle $\beta E \times \beta L$ is equal to the area intercepted by the ordinate βE , the curve EF, and the asymptote βC , infinitely produced. And the rectangle $\beta M \times \beta H$ is equal to the area intercepted by the ordinate βH , the curve HK, and the asymptote βC infinitely produced. Therefore these areas are equal. And these areas are as the pressures of all the atmosphere above B, in the temperatures to which the curves belong, respectively. Therefore the pressures in these different temperatures are equal. Q. E. D. I say further, that the pressure of the superincumbent atmosphere, at any height below B, is less in the greater temperature than in the cooler. Let AP be any height less than B, and take $Cp = \frac{CA^2}{CP}$, and draw the ordinate pN , cutting the curves in the points N and O. Now $pN, \beta E$ being as the densities at P and B, in the

the cooler condition of the air, and pO , βH as the densities at the same heights P and B, in the increased temperature, pO is less than to bear to βH , the proportion of pN to βE (by 4th of this): and, by permutation, pO is less than to bear to pN the proportion of βH to βE , or of βL to βM . Therefore $pO \times \beta M$ is less than $pN \times \beta L$; that is, the area intercepted between the ordinate pO , the curve OHK , and the asymptote pC infinitely produced, is less than the area intercepted by the ordinate pN , the curve NEF , and the asymptote pC infinitely produced. But again, these areas are as the pressures of all the atmosphere above P, in the temperatures to which the curves respectively belong; therefore, the pressures in the greater temperature, to which the curve OHK belongs, is less than the pressure in the cooler temperature, to which NEF belongs. *Q. E. D.* In like manner it may be shewn, that the pressures in the warmer temperature, at all heights above B, are greater than in the cooler.

And thus theory might have brought us to expect a phænomenon, which M. DE LUC hath actually observed, and was not a little surpris'd at. For if the temperature of the atmosphere be at any time gradually augmented, a barometer, placed below the height of stationary pressure, will sink, while another, placed higher up in the atmosphere, will rise. This is what M. DE LUC hath observed in two barometers at the foot and summit of a hill, in settled weather, while the natural heat of the day hath been upon the increase; from whence he, with
reason,

reason, concludes, that between these two stations, where the pressure was changing, at the same time, contrary ways, there must have been an intermediate one, which I call the height of stationary pressure, where no change, in either sense, could take place.

I shall hereafter shew, at what height the place of unaltered pressure should fall, by theory, for every change of temperature. It seems a problem worthy of a naturalist, to enquire how far theory doth, in this circumstance, agree with the real operations of nature.

6. It may seem, perhaps, still more surprizing, but it is no less true, that *there will generally be a particular height in the atmosphere where the density will remain unchanged, by a given change of temperature.* To determine in what changes this will happen, and at what height the place of unaltered density, for given changes of temperature, should fall, requires only the solution of the following problem.

PROBLEM FIRST.

To find the intersection of two logarithmics, which have a right line given in position for their common asymptote, and their subtangents given in magnitude; an ordinate in each curve, drawn at right angles with the common asymptote, through a given point in it, being also given in magnitude.

IMA-

IMAGINE two logarithmics, CDE, FDG (fig. 3, 4, 5, 6.), having the right line AB, given in position, their common asymptote; and suppose the subtangent of each curve given in magnitude. Through a given point A, in the common asymptote, imagine the right line AC drawn, an ordinate at right angles with the asymptote, meeting the curves in C and F; and let AC, AF be severally given in magnitude. It is required to find the point where these curves intersect. Suppose it done, and let D be the intersection. Draw DL perpendicular to AB. Through F, the point where AC meets one of the curves FDG, draw FM parallel to AB, meeting the other curve CDE in M. Draw MN perpendicular to AB, and take AH, AK equal to the given subtangents of the curves, CDE, FDG, respectively. Now AL is the logarithm of the *ratio* of AF to LD, in the system of the curve FDG; and (because NM = AF) NL is the logarithm of the same *ratio*, in the system of the curve CDE. Therefore, $AL : LN = AK : AH$. Therefore the proportion of AL to LN is given; and consequently, that of AN to AL is given. But AN is given in magnitude. For AC and AF are given in magnitude (by hypothesis). Therefore, the proportion of AC to AF or NM is given; and AN is the logarithm of that given proportion in the system of the given curve CDE. But AN being given in magnitude, and the proportion of AN to AL being given, AL is given in magnitude. And it is given in position, and the point A is given (by hypothesis). Therefore the point L is given (by 27. dat.). Therefore,
LD,

LD, being perpendicular to AL, is given in position (by 30. dat.). But AL being given in magnitude, the proportion of AF to LD is given (by logarithms). And AF is given in magnitude. Therefore LD is given in magnitude. Therefore the point D is given. *Q. E. I.*

The construction is obvious. It is evident, that the points L and N are on the same side of A, if F be at the greater curve, as in fig. 3 and 4; but on different sides of A, if the curve to which F belongs be the less, as in fig. 5 and 6 (*).

The calculation of the lengths AL, LD, by means of the logarithmic canon, is very simple. Putting B for the subtangent of the Briggsian system, L for the tabular logarithm of AC, and D for the difference of the tabular logarithms of AC, AF, we shall have,

$$\text{First, } AL = \frac{AH \times AK \times D}{B \times KH}.$$

$$\text{And again, } L \mp \frac{\overline{AK} \times \overline{D}}{KH} = \text{tab. log. of LD.}$$

In this second expression, the second term is negative, if the greater of the given ordinates belong to the less curve, as in fig. 3 and 5; but positive if the greater ordinate belong to the greater curve, as in fig. 4. and 6.

Both these theorems are so easily derived from the preceeding analysis of the problem, that it is needless to add the synthetic demonstration; but they may be reduced to more commodious forms for practice by the following artifice.

(*) By the greater and the less curve I mean that which hath the greater or the less subtangent.

First,

First, if AH be less than B, and AK greater,

put $AH = B - \frac{p}{q} B$, and $AK = B + \frac{s}{t} B$. Then $HK = \sqrt{\frac{s}{t} + \frac{p}{q}} B$.

And substituting these values for AH, AK, and HK, respectively, we shall have, by the theorem, for AL,

$$AL = \frac{qt + qs - tp - ps}{qs + tp} D.$$

And hence if $t = q$, $AL = \frac{q+s}{p+s} D \times 1 - \sqrt{\frac{p}{q}}$.

2. If AH and AK both exceed B,

put $AH = B + \frac{p}{q} B$, and $AK = B + \frac{s}{t} B$. Then $HK = \sqrt{\frac{s}{t} + \frac{p}{q}} B$.

In this case $AL = \frac{qt + qs + tp + ps}{qs - tp} D$.

And if $t = q$, $AL = \frac{q+s}{s-p} D \times 1 + \sqrt{\frac{p}{q}}$.

3. Again, if both AH and AK be less than B,

put $AH = B - \frac{p}{q} B$, and $AK = B - \frac{s}{t} B$. Then, $HK = \sqrt{\frac{p}{q} - \frac{s}{t}} B$.

And substituting these values, $AL = \frac{qt - qs - tp + ps}{tp - qs} D$; or if $t = q$,

$$AL = \frac{q-s}{p-s} D \times 1 - \sqrt{\frac{p}{q}}.$$

And, by the theorem, for LD, we shall have by due substitution, in the first case,

$$L = \frac{qs + qt}{qs + tp} D = \text{tab. log. LD}; \text{ or, if } t = q,$$

$$L = \frac{s+q}{s+p} D = \text{tab. log. LD}.$$

In the 2d case, $L = \frac{q^t + q^s}{q^s - t^p} D = \text{tab. log. LD}$; or, if $t = q$,

$$L = \frac{s + q}{s - p} D = \text{tab. log. LD}.$$

In the 3d case, $L = \frac{t - s \times q}{t^p - q^s} D = \text{tab. log. LD}$; or, if $t = q$,

$$L = \frac{q - s}{p - s} D = \text{tab. log. LD}.$$

The *formulae*, in which q and t are equal, will be of particular use in the application of this theory to the atmosphere, of which examples will shortly be given.

In Case 1, if $\frac{p}{q} = 0$; that is, if $AH = B$,

$$AL = D \times \sqrt{1 + \frac{t}{s}},$$

$$\text{and } L = D \times \sqrt{1 + \frac{t}{s}} = \text{tab. log. LD}.$$

$$\text{But if } \frac{s}{p} = 0, AL = D \times \sqrt{\frac{q}{p} - 1};$$

$$\text{and } L = D \times \sqrt{\frac{q}{p}} = \text{tab. log. LD}.$$

COR. *The intersection of two logarithmics being given, which have a right line given in position for their common asymptote, and subtangents severally given in magnitude; to find the point in the common asymptote, through which the ordinate, drawn at right angles with the asymptote, is cut by the curves in a given proportion, and to assign the magnitude of each segment, is the converse of the foregoing problem; and the same principles lead to its solution. For suppose A the point required. The proportion of AC to*

to AF being given (by hypothesis), the length of AL will be given, from the same analysis as before. But D being given, and the right line AB given in position, LD is given in position and magnitude; and, AL being given, the proportion of LD to AC, and also to AF, is given (by logarithms). Therefore AC and AF are each given.

The expression for the length of AL is the same as before. And $\text{tab. log. LD} \pm \frac{AK \times D}{KH} = L$.

In this expression the second term is positive, if the ordinate from A to the lesser curve is to be the greater of the two; in the contrary case, negative.

Now imagine CDE, FDG to be logarithmics of the atmosphere, in different temperatures; AC being the density at the earth's surface in one temperature, and AF in the other; and let AB be the semi-diameter of the earth: let the two logarithmics meet in D, and draw DL perpendicular to AB. Now if the point L be any where in the line AB, between A, which is at the surface of the earth, and B, which is the center, the ordinate LD will represent the density of the air, in the system of both curves, at the distance $\frac{BA \times AL}{BA - AL}$ above the earth's surface; and therefore, at this height, the density is the same in the one temperature as the other.

If L coincide with B, DL represents the density at an infinite height; but if L falls beyonds B, DL is not among the ordinates, of either curve, which represent densities any where existing. The expres-

sion $\frac{BA \times AL}{BA - AL}$, which is infinite when $AL = AB$,

now becomes more than infinite, the denominator being negative; and there is no height in the atmosphere, at which the density is the same in both temperatures. Again, if L fall above the earth's surface (as in fig. 4. and 6.), LD is not among the ordinates of either curve which represent densities;

and AL, being negative, the expression $\frac{BA \times AL}{BA - AL}$ be-

comes $-\frac{BA \times AL}{BA + AL}$; which expresses a distance below

the earth's surface; but whether L fall above or below the surface depends upon the state of the density at A, and the temperature jointly. If the density were to be greater, in the greater temperature, then the greater ordinate at A belongs to the greater curve (as in fig. 4. and 6.), and L is above the surface: but if the density be less, when the temperature is greater, then, of the two ordinates at A, the less belongs to the greater curve, and L is below the surface (as in fig. 3. and 5.); and in this case, the place of the point L depends upon the magnitude of AN, and the proportion of HK to AK. If $HK : AK = AN : AB$, then L and B (in fig. 3. and 5.) coincide, and the densities are the same at an infinite height. If HK be less than to bear to AK, the proportion of AN to AB, AL will be greater than AB, and L will fall beyond B, and the densities are no where the same. But if HK be greater than to bear to AK, the proportion of AN to AB, AL will be less than AB; and in this case there will

will be a height above the earth's surface, namely, $\frac{BA \times AL}{BA - AL}$, at which the densities, in both temperatures, will be the same. And it is evident, that this height is given, if AL be given. But AL is given, by the preceeding problem, if the subtangents AH, AK be given in magnitude, and the proportion of AC to AF be given. But if each temperature be given, each subtangent is given (by sect. 5. prob. 3.) and the proportion of AC to AF will be given by the barometer. (sect. 5. prob. 4.)

EXAMPLE.

| 1744, | a | b | Brs observed. | Tr in. | Tr out. | Brs reduced to common temperature | Densities, by §. 5. prob. 4. |
|----------|---|---------------|---------------|--------|---------|-----------------------------------|------------------------------|
| March 26 | 1 | $\frac{1}{2}$ | 30,17 | 54 | 61 | 30,176 | 137904 |
| April 2 | 6 | 6 | 29,37 | 56 | 48 | 29,37 | 138039 |

Hence $D = 0.0004080 = 4,08$ fathom.

But $p = 8$. $s = 21$. $q = 449$. $q + s = 470$. $s - p = 13$.

$$\text{and } \frac{q+s}{s-p} = 36,1.$$

Therefore by formula 2d, $AL = 150$ fathom; and at a height, insensibly greater, the density was the same in both constitutions of the atmosphere.

7. It is manifest, that the changes of density above and below the height where it remains unaltered, are contrary. If the lower densities are diminished, the higher ones are increased, and vice versâ. Notwithstanding the mathematical evidence of these conclusions, I am persuaded, it will appear to many a *physical* paradox,

paradox, that heat should any where *condense*. I have already hinted at the solution. Heat doth not condense any finite mass of matter, which is at liberty, on all sides, to expand itself; but acting on a finite mass, which hath not unlimited liberty of expansion, it may condense one part by the rarefaction of another. Acting on an infinite mass, it must do this; because it can only transpose. It neither generates new matter, nor annihilates what already is. What is taken therefore from one part, must be added to another, and *vice versa*; otherwise the quantity of matter, in the whole, must be changed.

Imagine ABCD to be a small portion of any orb of the atmosphere (fig. 8.) AEFB, CMND, CHGA, DLBK, contiguous portions. Heat drives many of the particles, which occupy the space ABCD, out of it; but it likewise drives out many of the particles which occupy the contiguous spaces. And of those which are driven out of the contiguous spaces, many will enter the space ABCD. If the particles which are driven *into* ABCD, be more in number than those which are driven *out* of it, the air, in this space, is condensed, by that very cause, which would rarefy it, if the contiguous portions were annihilated. Thus condensation in one part may, in an infinite fluid, must ensue from rarefaction in another, if the quantity of matter remains unaltered. Where then is the wonder, that the like effect should follow from the cause of rarefaction combined with other causes?

8. *The cases in which L falls below B, or above the surface, seem to be physically impossible, without*

an alteration of the quantity of matter in the atmosphere; for without this, it cannot be every where rarefied, or every where condensed at one and the same time.

9. *Having found the heights where the density is the same in any two different temperatures, the height where the pressures in different temperatures are equal, will easily be determined. For this purpose, we have only to seek the solution of the following problem..*

PROBLEM II.

Two logarithmics (DQF, GQK) intersecting in a given point (Q), (Vid. fig. 2.), having a right line (AC) given in position for their common asymptote, and subtangents severally given in magnitude; to find the point (β) in the asymptote; where an ordinate (β HE) being drawn at right angles, to meet both curves, the areas intercepted between the two curves and the common asymptote infinitely extended, beyond the ordinate β E, are equal.

LET β L, β M be equal to the subtangents of the curves DQF, GQK, respectively. Because the curvilinear areas are equal; therefore, the rectangles β L \times β E, β H \times β M, are equal. Therefore, β E : β H = β M : β L. But β M, β L, being given, the proportion of β M to β L is given. Therefore, the proportion of β E to β H is given. Therefore, the point β is given (by corollary of preceeding problem).

CONSTRUCTION.

Find the point β (by corollary of preceeding pre-problem) such that the ordinate $\beta H E$ being drawn, βE , βH , may be reciprocally as the subtangents ; or that βE may be to βH as the subtangent of the curve $G H K$ to that of $D E F$: and the thing propofed is done.

Now imagine the curves $G Q K$, $D Q F$, to represent the logarithmics corresponding to different given temperatures of the air, A being at the surface of the earth, and C at the center. And having found the point β , take $A B = \frac{A \beta \times C A}{C A - A \beta}$; then B is the place where the barometer will stand at the same height, in both conditions of the atmosphere.

LIMITS.

Draw the ordinates, $A G D$, $C F K$, and $Q R$. If the point Q be any where below the ordinate through A , it is evident, the point β will lye above the point R . And if the subtangents βL , βM , be reciprocally as the densities $A D$, $A G$, the points β , A , and consequently B , A , coincide, and the pressure, at the level of the sea, remains unchanged.

If the greater subtangent βM be greater than to bear to the less βL , the proportion of the greater density $A D$ to the less $A G$, β will fall above A , and B consequently will be below it; for $A \beta$ becoming
4
negative,

negative, the expression $\frac{CA \times A\beta}{CA - A\beta}$ becomes $-\frac{CA \times A\beta}{CA + A\beta}$

And there is no place in the atmosphere where the pressure is the same in one condition as the other. But it is, at all heights, greater in the warmer condition (by 5th of this).

If the point Q fall below the central ordinate CK, (fig. 7.) and the subtangents βL , βM , be reciprocally as CF, CK, the points β and C will coincide, and B will go off to an infinite height.

If the greater subtangent βM be less than to be βL as the greater density CF to the lesser CK, β will be found below C, and the ordinate βE is not one of those, by which the density of the air, at any height, in either condition, is represented.

The expression $\frac{CA \times A\beta}{CA - A\beta}$ is more than infinite, the denominator being negative; and there is no height at which the pressure is the same, in both conditions of the atmosphere; but it is at all heights less in the warmer condition (by 5th of this). But whenever Q falls below the superficial ordinate, whether it be above or below the central ordinate, if βM be less than to be to βL as AD to AG, and greater than to be to βL as CF to CK, β will fall between C and A, and an equality of pressure, in both conditions of the atmosphere, will take place at a finite height, determined as above.

If Q ever falls above the superficial ordinate through A, β will be more above it, and B will be below the earth's surface, and there will be no height at which the pressure will be the same;

but it will every where be greater in the warmer temperature.

The calculation for determining $A\beta$ is obvious from the foregoing solution of the problem. For putting B for the subtangent of the Briggian system, D for the difference of the tabular logarithms of AG , AD , and Δ for the difference of those of βL , βM , we have,

$$AR = \frac{\beta L \times \beta M \times D}{B \times LM} \text{ (by prob. 1.)}$$

$$\text{and } R\beta = \frac{\beta L \times \beta M \times \Delta}{B \times LM} \text{ (by cor. 1.)}$$

$$\text{Therefore, } A\beta = \overline{D - \Delta} \times \frac{\beta L \times \beta M}{B \times LM}.$$

Therefore, putting $\beta L = B \pm \frac{p}{q} B$, and $\beta M = B \pm \frac{s}{t} B$. and substituting $D - \Delta$ for D in the *formulae* for AL , deduced from problem 1, we shall change them into *formulae* for $A\beta$.

EXAMPLES.

In the example of the preceeding problem $D = 4,58$, and $\Delta = 121,817$ fathom. Whence $D - \Delta$ is negative, and the point B falls above the surface, and the pressure was no where the same, but at all heights greater in the warmer condition.

Fig. 1.

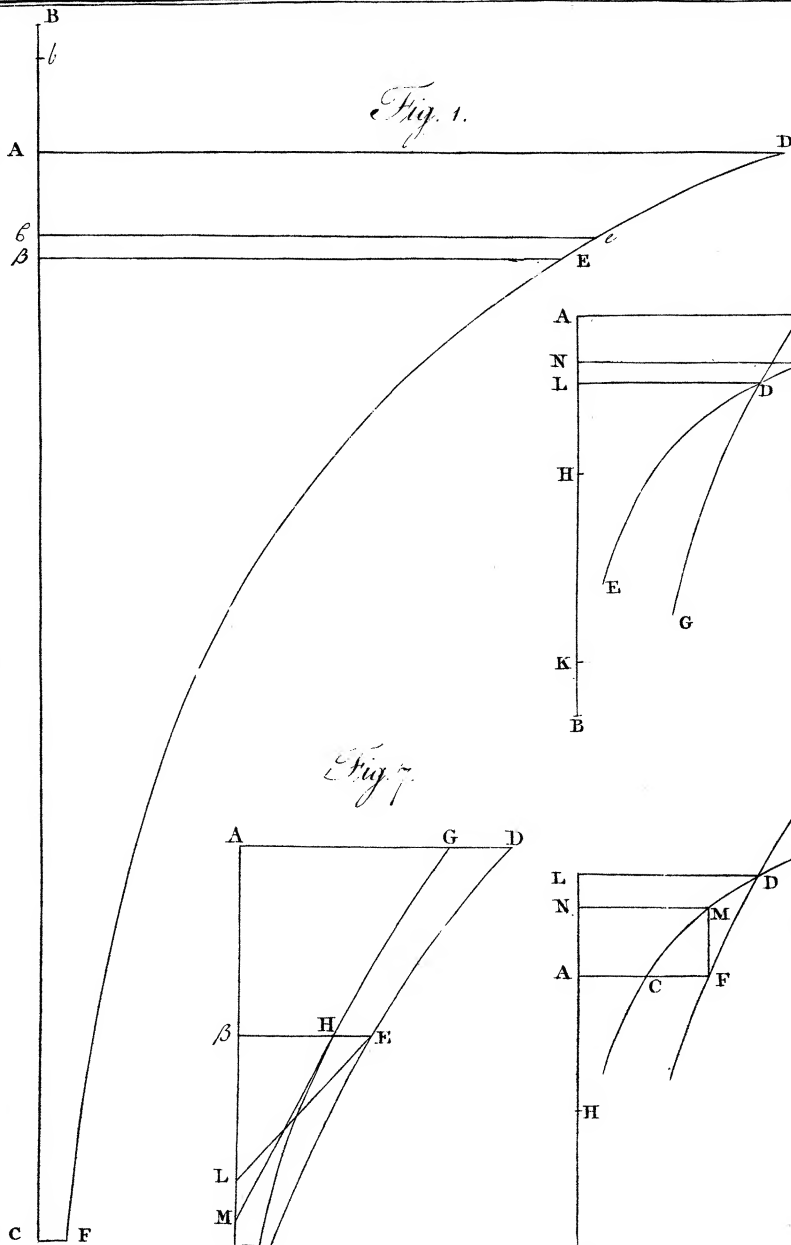


Fig. 3.

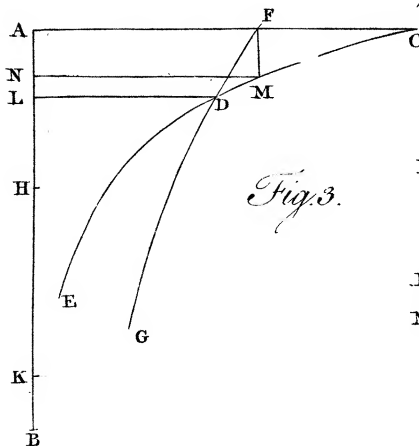
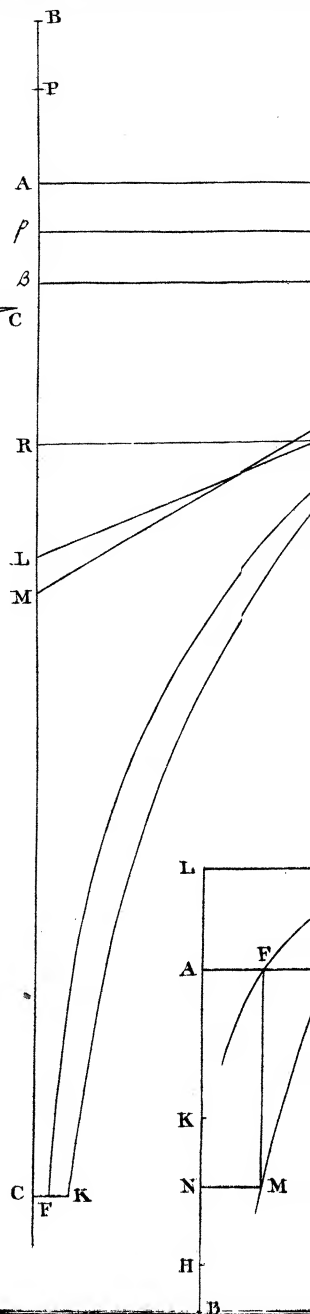
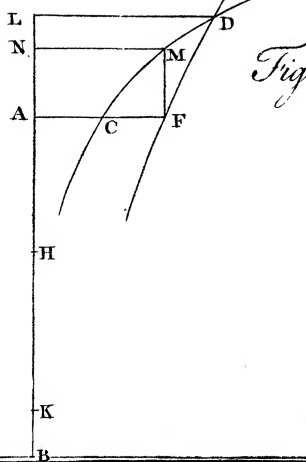
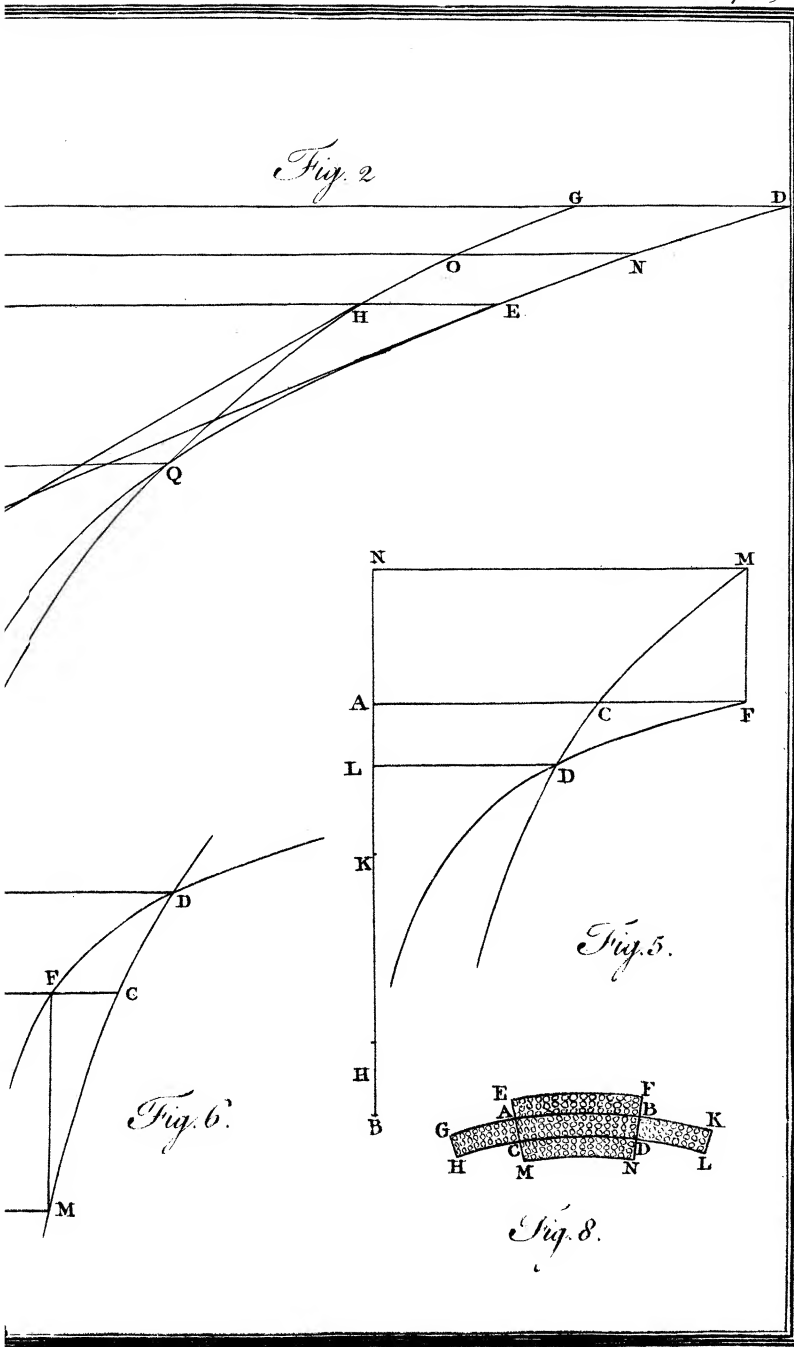


Fig. 4.





EXAMPLE II.

| d | h | B ^s . ob- served. | T ^r . in. | T ^r . out. | B ^r . reduced to common temperature | Densities, by §. 5. prob. 4. |
|----------|----|---------------------------------|-------------------------|--------------------------|--|---------------------------------|
| March 26 | 20 | 30,07 | 58 | 59 | 30,058 | 13796,6 |
| 28 | 21 | 30,21 | 54 | 50 | 30,21 | 14138,2 |

Hence $p=10$. $s=19$. $q=449$. $q+s=468$. $s-p=9$. $\frac{q+s}{s-p}=52$.

$D=0.0106238$. $\Delta=0.0084332$. $D-\Delta=0.0021906=21,906$ fathom. Therefore, $A\beta=1164,42$ fathom $=1,32$ miles; and at a height insensibly greater, the pressure was the same in both conditions.

In this manner may the heights be determined, in which either the density or pressure of the air, in a given temperature, is the same as in any greater or lower temperature; provided the proportion be known between the densities, in the different temperatures, at the level of the sea, or at a given elevation above it; otherwise both the problems are indeterminate. And I know no method of ascertaining this proportion, but by actual observation of the barometer. The change of temperature determines the proportion in which the subtangent of the atmospherical logistic is changed; but in whatever proportion the subtangent of a logistic is changed, the ordinate, at a given point of the asymptote, may be of any imaginable length, or may bear any imaginable proportion to the subtangent. There is no given *geometrical* relation, in the nature of the curve, between the length of the one and the other; and experiment hath not yet brought to light any *physical* relation in the particular case in question. Diligent observation of the barometer and ther-

mometer may perhaps, in future, give more satisfactory information upon the subject. Perhaps I have been too minute in detailing consequences from an hypothesis, of what probably never doth obtain; namely, that the atmosphere may be considered as equally heated in all its different parts; and that the variation of the absolute elasticity of its component particles is proportional to the change in the degree of uniform temperature, as if the absolute elasticity were influenced by no other cause. But such is the necessary order of enquiry. Theory must begin with the most simple cases, considering what *would* be the effect of some known cause, acting singly, and in the most simple manner; and comparing the conclusions from what it is *supposed*, with the effects which *are*, the *difference* leads us to the discovery of other causes, and to an estimation of the separate effects of each, and of the compound effects of all the known causes any how combined. I shall only add, that as the state of an atmosphere, unequally heated in its different parts, may be represented by parts of different logarithmics, it is possible, that instead of one point of unaltered density, and one point of unaltered pressure, we might find several, if synchronous observations could be made at several successive elevations sufficiently different.

10. If the atmosphere of the earth reaches to infinite heights with a finite density; for the same reasons, that of Jupiter and every other planet, will reach also to infinite heights, above the surface of the planet with a finite density. The atmosphere therefore of every planet will reach to the surface of every other planet, and to the surface of the Sun;

 and

and the atmosphere of the Sun to the surfaces of them all. All these atmospheres will mingle, and form a common atmosphere of the whole system. This common atmosphere of the system will be infinitely diffused, since the particular atmospheres that compose it are so. It will reach therefore to every fixed star; and, for the same reason, that of every fixed star will reach the central body of our system, and of every other system. The atmospheres of all the systems will mix. The universe will have one common atmosphere, a subtle elastic fluid, which pervades infinite space, and being condensed near the surface of every larger mass of matter, by the gravitation towards that mass, forms its peculiar atmosphere.

To certain distances from every one of these great bodies, the condensations of this infinite fluid will follow the laws of the preceeding theory *nearly*; but to certain distances *only*. For that theory considers only the effects of the attraction of a single sphere, and assigns the law of the variation of density, such as would obtain, if *one* spherical body existed in the midst of an infinite fluid; and such as cannot generally obtain, unless that hypothesis were true: for many great spheres being immersed in a common atmosphere, the attraction of any one, at great distances from it, becomes but an inconsiderable part of the whole cause, by which the density is modified, the joint forces of them all. And in many other circumstances, besides the condensation, the modifications of every particular atmosphere may depend upon those of others, innumerable and infinitely remote; as the *effluvia* and exhalations of each

each great mass, mingling with its atmosphere, may be distributed, in certain proportions, among all the rest. So that there is probably no branch of physics, in which human discovery, in its utmost extent, must always bear so small a proportion to what will still remain unknown.

T A B L E I.
EQUATION of the boiling point.

| Barometer. | Equation. | Difference. |
|------------|-----------|-------------|
| 31,0 | + 1,57 | |
| 30,5 | + 0,79 | 0,78 |
| 30,0 | 0,00 | 0,79 |
| 29,5 | — 0,80 | 0,80 |
| 29,0 | — 1,62 | 0,82 |
| 28,5 | — 2,45 | 0,83 |
| 28,0 | — 3,31 | 0,85 |
| 27,5 | — 4,16 | 0,86 |
| 27,0 | — 5,04 | 0,88 |

The numbers in the first column to the left express heights of the quicksilver in the Barometer, in English inches and decimal parts. The second column gives the Equation to be applied, according to the sign prefixed, to 212° of Bird's Fahrenheit, to find the true boiling point, for every such state of the barometer. The heights of the barometer decrease by $\frac{1}{2}$ inches from 31 to 27 inches. The boiling point, for all intermediate states of the barometer, may be found, with a very sufficient accuracy, by taking proportional parts. For which purpose, the differences of the equations are given in the third column.

TABLE II.
For the Comparison of Thermometers.

| | LF | PF |
|---|--------|--------|
| 1 | 2,225 | 2,246 |
| 2 | 4,450 | 4,493 |
| 3 | 6,674 | 6,74 |
| 4 | 8,9 | 8,986 |
| 5 | 11,124 | 11,233 |

| | L reduced. | P reduced. |
|------|------------|------------|
| 80 | 210,0 — | 211,73 — |
| 75 | 198,86 + | 200,50 — |
| 70 | 187,74 + | 189,26 + |
| 65 | 176,62 — | 178,03 — |
| 60 | 165,49 + | 166,80 — |
| 55 | 154,37 — | 155,56 + |
| 50 | 143,24 + | 144,33 — |
| 45 | 132,12 + | 133,09 — |
| 40 | 121,0 — | 121,85 + |
| 35 | 109,87 + | 110,62 + |
| 30 | 98,75 — | 99,39 — |
| 25 | 87,62 + | 88,15 + |
| 20 | 76,50 + | 76,92 + |
| 15 | 65,38 — | 65,69 — |
| 10 | 54,25 + | 54,45 + |
| 5 | 43,13 — | 43,22 + |
| 0 | 32. | 32. |
| — 5 | 20,88 — | 20,77 — |
| — 10 | 9,75 + | 9,53 + |
| — 15 | 1,37 + | 1,70 — |
| — 20 | 12,50 — | 12,93 + |
| — 25 | 23,62 — | 24,16 + |
| — 30 | 34,74 + | 35,40 — |
| — 35 | 45,87 — | 46,63 + |
| — 40 | 57,00 — | 57,86 + |
| — 45 | 68,12 — | 69,10 — |
| — 50 | 79,24 — | 80,33 — |
| — 55 | 90,36 + | 91,56 + |
| — 60 | 101,49 — | 102,80 — |
| — 65 | 112,61 + | 114,03 — |
| — 70 | 123,75 — | 125,25 + |
| — 75 | 134,87 — | 136,48 + |
| — 80 | 146,0 — | 147,72 — |

| | FL | FP |
|----|-------|-------|
| 1 | 0,449 | 0,445 |
| 2 | 0,899 | 0,890 |
| 3 | 1,348 | 1,335 |
| 4 | 1,798 | 1,780 |
| 5 | 2,247 | 2,225 |
| 6 | 2,696 | 2,671 |
| 7 | 3,146 | 3,116 |
| 8 | 3,595 | 3,561 |
| 9 | 4,045 | 4,006 |
| 10 | 4,494 | 4,451 |

EXPLANATION.

The principal table consists of three columns. The numbers in the first column signify degrees, either of DE LUC's scale, or the scale which hath been hitherto chiefly in use among the French, which is divided as M. DE LUC's is; but 28 French inches being generally mentioned, by the French mathematicians, as the mean height of the barometer at PARIS, it is to be supposed, that their boiling point will generally agree to that state of the barometer. The numbers in the next column to the right, which is marked *L reduced*, give the degrees of BIRD's Fahrenheit corresponding to the degrees of M. DE LUC's scale, expressed by the numbers of the first column; and the numbers in the column marked *P reduced*, give the degrees of BIRD's Fahrenheit corresponding in like manner with the PARIS scale. The heights in the first column diminish by 5° successively. But the reduction may be computed to every degree, by means of the little table on the left; which gives the value of single degrees, either of DE LUC's, or of the PARIS scale, in degrees of BIRD's Fahrenheit. The titles LF, PF signifying DE LUC's degrees, or Paris degrees, reduced to Fahrenheit's, respectively.

If there should be occasion to reduce heights of BIRD's Fahrenheit, either to M. DE LUC's, or the PARIS scale, this may be done by the principal table, and the small one on the right hand; which exhibits the value of single degrees of BIRD's Fahrenheit in degrees, both of DE LUC's and the Paris scales: the titles FL, FP signifying degrees of BIRD's Fahrenheit reduced to M. DE LUC's degrees, or the PARIS degrees, respectively.

EXAMPLE.

To find the point upon a BIRD's Fahrenheit, which corresponds to + 58 of DE LUC's scale.

| | |
|----------------------------------|---------------|
| By principal table | + 55 = 154,37 |
| By little table on the left hand | 3 = 6,67 |
| Therefore | + 58 = 161,04 |

EXAMPLE 2.

To find the point upon the Paris scale, which corresponds to + 55 of BIRD's Fahrenheit.

| | |
|-----------------------------------|--------------|
| By principal table | + 54,45 = 10 |
| By little table on the right hand | 0,5 = 0,22 |
| By ditto | 0,05 = 0,02 |
| Therefore | + 55 = 10,24 |

T A B L E

T A B L E I I I.

EQUATION for the temperature of the QUICKSILVER.

| Degrees of Bird's Fahrenh. | Correction of Diff. Logarith. | Fathom | |
|----------------------------------|-------------------------------------|--------|-------|
| 1 | 452 | 0,452 | 0,003 |
| 2 | 904 | 0,904 | 0,006 |
| 3 | 1355 | 1,355 | 0,009 |
| 4 | 1807 | 1,807 | 0,012 |
| 5 | 2259 | 2,259 | 0,016 |
| 6 | 2711 | 2,711 | 0,019 |
| 7 | 3163 | 3,163 | 0,022 |
| 8 | 3614 | 3,614 | 0,025 |
| 9 | 4066 | 4,066 | 0,028 |
| 10 | 4518 | 4,518 | 0,031 |
| 20 | 9036 | 9,036 | 0,062 |
| 30 | 13554 | 13,554 | 0,094 |
| 40 | 18072 | 18,072 | 0,125 |
| 50 | 22590 | 22,590 | 0,156 |
| 60 | 27108 | 27,108 | 0,188 |
| 70 | 31626 | 31,626 | 0,219 |

The numbers in the first column are to be understood to express differences of temperature, in the quicksilver of the portable Barometers at the two stations, indicated by Thermometers fixed in the cases of the Barometers, in degrees of Bird's Fahrenheit. The second column gives the corresponding corrections of the difference of the tabular logarithms of the observed heights of the quicksilver in the barometer, to be added or subtracted, according as the higher barometer hath been the warmer or cooler of the two. The numbers in the third column are the $\frac{1000}{1000}$ th parts of those in the second. The manner of using them is explained in the general precepts. The

fourth column gives the reduction of the observed height of the column of quicksilver, in decimal parts of an inch, for the differences of temperature expressed by the numbers of the first column, when the height of the quicksilver is 30 inches; and the reduction for any other height of the quicksilver may be thus found. Suppose the height of the barometer $30 \pm a$ inches, in a certain temperature b , and this is to be reduced to the temperature $b \pm n$. From the fourth column collect the number corresponding to n (by repeated entry, if need be). Call that number M. Then $M \pm \frac{a}{30} M$ is the reduction.

TABLE IV. EQUATION for the Temperature of the AIR.

| | 40 + | 50 + | 60 + | 70 + | 80 + |
|-----|------|--------|--------|--------|--------|
| 1 | 0,0 | 0,022 | 0,044 | 0,067 | 0,089 |
| 2 | 0,0 | 0,044 | 0,089 | 0,134 | 0,178 |
| 3 | 0,0 | 0,067 | 0,134 | 0,200 | 0,267 |
| 4 | 0,0 | 0,089 | 0,178 | 0,267 | 0,356 |
| 5 | 0,0 | 0,111 | 0,223 | 0,334 | 0,445 |
| 6 | 0,0 | 0,134 | 0,267 | 0,401 | 0,535 |
| 7 | 0,0 | 0,156 | 0,312 | 0,468 | 0,624 |
| 8 | 0,0 | 0,178 | 0,356 | 0,535 | 0,713 |
| 9 | 0,0 | 0,200 | 0,401 | 0,601 | 0,802 |
| 10 | 0,0 | 0,223 | 0,445 | 0,668 | 0,891 |
| 20 | 0,0 | 0,445 | 0,891 | 1,337 | 1,782 |
| 30 | 0,0 | 0,668 | 1,337 | 2,005 | 2,673 |
| 40 | 0,0 | 0,891 | 1,782 | 2,673 | 3,564 |
| 50 | 0,0 | 1,114 | 2,228 | 3,342 | 4,455 |
| 60 | 0,0 | 1,337 | 2,673 | 4,010 | 5,347 |
| 70 | 0,0 | 1,559 | 3,119 | 4,678 | 6,238 |
| 80 | 0,0 | 1,782 | 3,564 | 5,347 | 7,129 |
| 90 | 0,0 | 2,005 | 4,010 | 6,015 | 8,020 |
| 100 | 0,0 | 2,228 | 4,455 | 6,683 | 8,911 |
| 200 | 0,0 | 4,455 | 8,911 | 13,367 | 17,822 |
| 300 | 0,0 | 6,683 | 13,367 | 20,050 | 26,733 |
| 400 | 0,0 | 8,911 | 17,822 | 26,733 | 35,645 |
| 500 | 0,0 | 11,139 | 22,278 | 33,417 | 44,556 |
| 600 | 0,0 | 13,367 | 26,733 | 40,100 | 53,467 |
| 700 | 0,0 | 15,594 | 31,189 | 46,783 | 62,378 |
| 800 | 0,0 | 17,822 | 35,645 | 53,467 | 71,289 |
| 900 | 0,0 | 20,050 | 40,100 | 60,150 | 80,200 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0,002 | 0,004 | 0,007 | 0,009 | 0,011 | 0,013 | 0,015 | 0,018 | 0,020 |
| 2 | | 0,009 | 0,013 | 0,018 | 0,022 | 0,027 | 0,031 | 0,036 | 0,040 |
| 3 | | | 0,020 | 0,027 | 0,033 | 0,040 | 0,047 | 0,053 | 0,060 |
| 4 | | | | 0,036 | 0,044 | 0,053 | 0,062 | 0,071 | 0,080 |
| 5 | | | | | 0,055 | 0,067 | 0,078 | 0,089 | 0,100 |
| 6 | | | | | | 0,080 | 0,093 | 0,107 | 0,120 |
| 7 | | | | | | | 0,109 | 0,125 | 0,140 |
| 8 | | | | | | | | 0,142 | 0,160 |
| 9 | | | | | | | | | 0,180 |

In the principal Table, the numbers in the first column to the left express fathom. The numbers at the top of each vertical column express degrees of Bird's Fahrenheit above the point 0. The numbers under these, in every column, shew the corrections corresponding to these temperatures, respectively, in fathoms and decimal parts, upon so many fathom, in an approximate determination of the height in question, as are expressed by the numbers in the first column to the left. The little Table gives the correction for single degrees of Bird's Fahrenheit. The numbers in the uppermost horizontal row, and the outermost vertical row, to the left, expressing, the one fathom, and the other degrees of the thermometer, in-
differently.

GENERAL PRECEPTS, for the CALCULATION of HEIGHTS by
these TABLES.

FROM the common tables of logarithms, namely, those which exhibit the Briggian logarithms to eight places, write out the logarithms of the numbers, which express the observed heights of the quicksilver, in the portable barometers, at the two stations.

2. Subtract the lesser logarithm from the greater.

3. Divide the remainder by 1000. The quotient is a certain number of fathom.

4. Take the difference of the temperatures of the quicksilver, as indicated by the thermometers in the cases of your portable barometers, and look for the correction corresponding thereto, in the third column of TABLE III. and add that correction to the number last found, if the barometer at the higher station hath been the warmer of the two; otherwise subtract it. Call the sum in the first case, the remainder in the latter, the *approximate height*.

5. Add together the temperatures expressed at the two stations by the thermometers in the open air, under their proper signs. Take half the aggregate, and call it the *temperature of the air*.

6. In the principal table of the equation for the temperature of the air (TABLE IV.), look a-top for the decade of degrees next less than the temperature of the air, found by the last rule; and from the column underneath it (by repeated entries if need be) collect the corrections for the units, decades, and centuries of

fathom in the approximate height ; and if you have chiliads of fathom in your approximate height, find the correction for them, by taking the decuple of the correction for the corresponding centuries. Add these several parts of the correction together, and call the sum N .

7. In the little table look, either a-top or in the outer vertical row to the left, for the units of degrees in the temperature of the air, over and above decades, and collect the correction for these, for the units, decades, centuries, &c. of fathom in the approximate height, by taking the numbers found in the little table, and proper multiples thereof. Add these numbers together, and call the sum n .

8. If the sign of the correction found in the great table be $+$, add $N + n$ to the approximate height ; but if the sign of the correction found in the great table be $-$, subtract $N - n$ from the approximate height. The sum in the former case, the remainder in the latter, is the correct height, according to M. DE LUC's rules.

EXAMPLE.

EXAMPLE.

| Barometer. | Tr. in. | Diff. | Tr. out. | Tempe- rature of air. | |
|---|------------|-------|-------------|-----------------------------|---|
| Lower 29,68 | +57 | | +57 | +45½ | Diff. of Thermometers in = 14° = 10° + 4°. 10° gives 4,519; 4° gives 1,808. |
| Upper 25,28 | +43 | 14 | +43 | | |
| Log. of lower B ^r . of upper. | | | | 14724639 14027771 | Equation for $\frac{1}{2}$ 6,327 |
| Diff. 1000 | | | | 696,868 | Temperature of air = 49½° = 40° + 9°½. N = + 0 |
| Equation for $\frac{1}{2}$ | | | | -6,327 | $\left. \begin{array}{l} 9^\circ \times 600 \text{ fathom gives } 12 \\ 9^\circ \times 90 \quad \quad \quad 1,8 \\ \frac{1}{2} \times 600 \quad \quad \quad 0,6 \\ \frac{1}{2} \times 90 \quad \quad \quad 0,1 \end{array} \right\} \text{ by little table for air.}$ |
| Approximate height | | | | 690,541 | |
| Equation for air N + n | | | | + 14,5 | |
| Height correct | | | | 705,04 | |
| | | | | | $n = 14,5$ N + n = 14,5 |

EXAMPLE II.

| Barometer. | Tr. in. | Diff. | Tr. out. | Tempe- rature of air. | |
|----------------------------|------------|-------|-------------|-----------------------------|---|
| Lower 29,45 | +38 | | +31 | 33 | Diff. of Tr ^s . in = 3° gives equation for $\frac{1}{2}$ 1,336. Temperature of air = 33° = 30° + 3°. 30° upon 300 fathom gives 6,683 } by great ta- upon 90 2,005 } ble for air. upon 6 0,134 } sign — |
| Upper 26,82 | +41 | 3 | +35 | | |
| Log. of lower of upper | | | | 1.4690853 1.4295908 | N = 8,822 |
| Diff. 1000 | | | | 394,945 | $\left. \begin{array}{l} 3^\circ \times 300 \text{ fathom gives } 2,000 \\ \times 90 \quad \quad \quad 0,600 \\ \times 6 \quad \quad \quad 0,04 \end{array} \right\} \text{ by little table for air.}$ |
| Equation for $\frac{1}{2}$ | | | | + 1,356 | |
| Approximate height | | | | 396,301 | |
| Equation for Air — N — n. | | | | - 6,182 | |
| Height correct | | | | 390,119 | $n = 2,640$ N — n = 6,182 |

